Special Relativity I: Spacetime

Basic Definitions and Postulates

• *Event, spacetime manifold:* Spacetime is the set of all possible locations of physical phenomena localized in space and time, or *events*. In ordinary approaches spacetime is modeled as a 4-dimensional manifold M, a set of elements with suitable sets of coordinate systems used to identify events.

• *Inertial coordinates:* An inertial coordinate system $\{x^\mu\} = \{(t, \mathbf{x})\}$ on M is one based on an inertial observer. Conventionally $\mu = 0, 1, 2, 3$, and $x^0 = t$ is time measured by a clock carried by the inertial observer. The location (worldline) of that observer is assumed to be at $\mathbf{x} = 0$. Here we will usually assume we are using Cartesian coordinates. Coordinate systems of other types (spherical, cylindrical, etc) are related to Cartesian ones (with the same t axis) by the usual coordinate transformations.

• *Postulate 1:* (Relativity) The laws of physics are the same in all inertial reference frames. This principle was in effect used in prerelativity physics, but it gained prominence only with special relativity.

• *Postulate 2:* (Speed of light) The speed of light has the same value in all reference frames. This fact is highly non-intuitive, but it follows from the results of the Michelson-Morley experiment.

• *Causality:* Because the speed of light is independent of the source, each event $p \in M$ has an absolute past (the set of events that can influence p) and a future (the set of events that p can influence). The boundaries of those sets are the light cone of p, representing events that can be reached moving towards/from p at the speed of light. Knowing the light cones of all $p \in M$ is equivalent to knowing the causal structure of M.

• *Worldline:* A 1D line in M representing the set of locations of a particle in time. Because particles cannot move faster than light, each point along a worldline must be in the past or future of every other one.

• *Coordinates, space, simultaneity:* Coordinate systems (when not considering gravity and curved spacetime) ordinarily consist of one timelike coordinate t whose value represents the reading of a clock at each location, and three spacelike coordinates $\mathbf{x}$. Space is the set of all events in M with the same value of t; the geometry of such a surface is the usual Euclidean one, and different events in it are called simultaneous.

Symmetries of Spacetime

• *Poincaré trasformations:* The most general linear maps $\Lambda : M \to M$ that take inertial frames to other inertial frames, consistently with the speed of light postulate. The simplest ones are translations $x^\mu \mapsto x^\mu + a^\mu$, where the a^μ are constants. The remaining ones, rotations and boosts, are Lorentz trasformations.

• *Rotations:* Rotations can be represented by matrices R such that $\mathbf{x} \mapsto \mathbf{x}' = R\,\mathbf{x}$, as in Euclidean geometry. For example, for a rotation by an angle θ around the z axis the spatial coordinates transform according to

$$R = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

In Euclidean geometry the squared length of a vector $\mathbf{x}$ is given by $\|\mathbf{x}\|^2 := \mathbf{x}\cdot\mathbf{x}$, which can also be written in matrix form as $\mathbf{x}^{\mathrm{T}}\,1\,\mathbf{x}$, and the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ can be obtained from the dot product $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^{\mathrm{T}}\,1\,\mathbf{y}$, where here "1" is the identity matrix. Since a rotation $\mathbf{x} \mapsto R\,\mathbf{x}$ (under which $\mathbf{x}^{\mathrm{T}} \mapsto \mathbf{x}^{\mathrm{T}} R^{\mathrm{T}}$) does not change lengths or angles, the matrix R must be such that

$$R^{\mathrm{T}} 1\, R = R^{\mathrm{T}} R = 1\ , \quad \text{i.e.}\ , \quad R^{\mathrm{T}} = R^{-1}\ (R \text{ is an } \textit{orthogonal} \text{ matrix})\ .$$

If we take the determinant of the equality $R^{\mathrm{T}} 1\, R = 1$ ("R preserves the matrix 1") and use $\det R = \det R^{\mathrm{T}}$, we get $(\det R)^2 = 1$. A matrix with $\det R = -1$ would invert the orientation of the axes, so we choose $\det R = 1$ (R is a *special* matrix). Because rotations are 3×3 matrices, we end up with the group SO(3).

• *Boosts:* Transformations between frames set up by observers with non-zero relative velocities. For example, if observer O' is moving with velocity $\mathbf{v} = v\,\mathbf{i}$ with respect to observer O, then $x'^\mu = \Lambda^\mu{}_\nu\, x^\nu$, where

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \qquad \gamma := (1-\beta^2)^{-1/2} , \qquad \beta := v/c .$$

• *Invariant interval:* For any $\Delta x^\mu = x'^\mu - x^\mu$, all Poincaré transformations leave invariant the expression

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 ,$$

which therefore represents a coordinate independent, physically meaningful quantity characterizing the relationship between two events. *Interpretation:* Assume for definiteness that $t < t'$; then (i) if $(\Delta s)^2 = 0$ (null-related events) a signal traveling at the speed of light can travel from x to x'; (ii) if $(\Delta s)^2 < 0$ (timelike-related events), a signal slower than light can reach x' from x, there exists a reference frame in which the two events occur at the same spatial location, and $\sqrt{-(\Delta s)^2} = \Delta\tau$ is the proper time elapsed in that frame; (iii) if $(\Delta s)^2 > 0$ (spacelike-related events) there is a reference frame in which the two events are simultaneous and $\sqrt{(\Delta s)^2} = L_0$ is their (proper) distance in that frame.

Note: We will usually assume that in our units the numerical value of the speed of light is 1; for example, we might be measuring times in seconds and distances in light-seconds.

Lorentz Trasformations and Minkowski Spacetime

• *Minkowski spacetime and metric:* Minkowski spacetime is the model for spacetime in special relativity, a 4D space (an affine manifold) M topologically equivalent to $\mathbf{R}^4$, equipped with the Minkowski metric $\eta_{\mu\nu}$, given in the usual coordinates by the matrix

$$\eta = \mathrm{diag}(-1, 1, 1, 1) ,$$

and used to calculate intervals as

$$(\Delta s)^2 = \Delta x^\mu\, \eta_{\mu\nu}\, \Delta x^\nu = \Delta x^{\mathrm{T}} \eta\, \Delta x .$$

• *Lorentz group:* Lorentz transformations are such that

$$\Delta x^{\mathrm{T}} \Lambda^{\mathrm{T}} \eta\, \Lambda\, \Delta x = \Delta x^{\mathrm{T}} \eta\, \Delta x \quad \text{for all} \quad \Delta x^\mu , \qquad \text{or} \qquad \Lambda^{\mathrm{T}} \eta\, \Lambda = \eta .$$

Thus, Lorentz transformations are those preserving the 4×4 diagonal matrix η with one eigenvalue of the opposite sign, i.e., the special orthogonal group SO(3,1).

• *Spacetime geometry:* All geometrical quantities related to M can be calculated using the metric. For example, the length of a spacelike curve $\gamma : \mathrm{I} \to M$ that maps $\lambda \mapsto x^\mu(\lambda)$ between events A and B is

$$L = \int_A^B \mathrm{d}\lambda \sqrt{\eta_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} .$$

⋄ *Other related concepts:* Be able to represent on a spacetime diagram spacetime subsets and transformations, the relativity of simultaneity, proper time and time dilation, proper length and length contraction, the twin paradox. Example: The car and garage paradox.

Reading

Our textbook: Carroll, Chapter 1 up to Section 3; *Other books:* Wald, Ch 2; Schutz, Ch 2.