Superfluidity

Objective:

The objective of this report is to understand superfluidity and the science and circumstances under which it is achieved. In order to achieve this goal in section 1 background information is provided on the condensation of bosons into their lowest energy orbital. Then in section 2 connections are made between Bose-Einstein Condensation and superfluidiy. insection 3 the zero viscosity observed in superfluids is discussed , section 4 touches on the theoretical and experimental values needed for the occurrence of superfluidity, section 5 briefly mentions the occurrence of superfluidity in a fermionic system and section 6 concludes how Bose-Einstein condensation is connected to superfluidity.

1-Background

The study of superfluidity is best understood by understanding Bose-Einstein Condensation. To do so one can utilize the grand canonical ensemble and consider a system of bosons with fixed number of particles N in which N is a function of the chemical potential μ. One knows that thermodynamic variables are interconnected in a way that if a thermodynamic variable like temperature varies, μ might change as well. Therefore in order for the number of particles to remain fixed one should consider how μ and temperature T are interconnected by looking at the $\beta(\varepsilon - \mu)$ factor appearing in relation (1) which describes the total number of particles for a Bose-Einstein distribution.

$$N = \int_0^\infty d\varepsilon g(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu)}-1} \quad (1)$$

Where $\beta=\frac{1}{k_bT}$, ε is the energy of an orbital and k_b is the Boltzmann constant.

In order to keep N fixed, it is inferred from $\beta(\varepsilon - \mu)$ that as T increases μ needs to approach ε .. On the other hand if T decreases there will be a temperature T_0 where $\mu \simeq 0$ and by replacing $\mu \simeq 0$ in equation (1) and evaluating the integral the following relation will be obtained:

$$N = \zeta(\tfrac{3}{2})g_s n_Q^0 V \quad (2)$$

Where $\zeta(\frac{3}{2}) \simeq 2.612$,and n_Q^0 is the quantum concentration at T_0 defined by $n_Q^0 = (\frac{m\, k_b T_0}{2\pi\hbar^2})^{\frac{3}{2}}$.

Relations (1) and (2) and the fact that $\mu < \varepsilon$ raise the question: what will happen to N for temperatures below T_0. This question will lead to the realization that at low enough temperatures , where most bosons reside on the lowest energy orbital, the continuum approach for evaluating the integral in relation (1) will yield a different value for the number of particles, and therefore it will undermine the consistency of N. To overcome this problem, for T<T_0 one could include the average occupation of bosons for the lowest-energy orbital $< N_0 >$ separate from the integral observed in relation (1) and then use the continuum approach for the higher energy levels. This correction will present the following relation for the total number of particles:

$$N =< N_0 > \quad + \zeta(\tfrac{3}{2}) g_s n_Q V \quad (3) ,$$

where $n_Q = (\frac{m k_b T}{2\pi\hbar})^{\frac{3}{2}}$. Now if relation (3) is rearranged by using an expression for N in terms of T_0 [obtained from relation (2)] the following relation will be obtained for the fraction of particles in the lowest energy orbital

$$\frac{<N_0>}{N} =1 - (\frac{T}{T_0})^{\frac{3}{2}} \ . \ (4)$$

Relation (4) delineates the condensation in a Bose-Einstein distribution. This relation shows that as temperatures fall below T_0 most bosons position themselves on the lowest energy orbital to the point where at temperature T = 0 the entirety of the bosons will reside on the lowest energy orbital. Moreover one can rearrange relation (2) and find the critical temperature of Bose Einstein condensation T_0 as the following:

$$T_0 = \frac{2\pi\hbar^2}{m k_b} \left(\frac{n}{\xi(\frac{3}{2}) g_s} \right)^{2/3} \ . \ (5)$$

2-Superfluidity

Experimental observations show that at *T* = 2.176 K, liquid He_4 goes to a state in which its viscosity is zero. In other words it becomes a superfluid. Interestingly enough the T_0 obtained by equation (5) will return a value of 3.13 K for He_4. Hence It should come as no surprise if one realizes the connection between superfluidity and Bose-Einstein condensation. This connection will even make more sense if the interactions between helium atoms which can not be neglected are considered. In other words it would be safe to assume the difference between $T_0 = 3.13$ K and the experimental value of *T*= 2.176 K arises from the assumption that bosons are non interacting particles.

3-Lack of Viscosity

In order to better comprehend the concept of zero viscosity it is beneficial to think of a massive particle moving within a fluid with some velocity. In regular circumstances the particle will collide with fluid particles and cause rarefaction and contraction within fluid particles and hence through the propagation of these contractions and rarefactions longitudinal waves will form and therefore transfer energy and momentum from the object to the fluid. In return this will lead to the object experiencing a retarding viscous force which it would not have experienced if the fluid was replaced by a superfluid. In a superfluid the quanta of the longitudinal waves are elementary excitations of the liquid known as phonons. Unless the object moves faster than a certain critical speed, it will not experience a viscous force which is due to the nature of the excitations in a superfluid.

To better clarify this concept it would be good to assume liquid He_4 at zero temperature which will soon have an object with mass M_0 and velocity V_0 added to it. Upholding the conservation of energy means the introduction of such an object will cause the excitation of He_4 in a way that there will be a density of excitation that possesses the transferred momentum of ħk and energy of ε_k(transferred from the particle introduced to the liquid He_4). Furthermore, other than conservation of energy one needs to take conservation of momentum into account as well. So we may write:

$$\frac{1}{2}M_0V_0^2 = \frac{1}{2}M_0V_1^2 + \varepsilon_k \qquad (6)$$

$$M_0V_0 = M_0V_1 + \hbar k \qquad (7)$$

where V_1 is the velocity of our object after the excitation of the liquid.

A legitimate question on the matter would be how to connect the absence of viscosity with equations (6) and (7). The point is that both equations cannot be satisfied simultaneously for the form of energy excitations found in liquid He_4. If one takes the square of equation (7) and rearranges it to obtain the following

$$\frac{1}{2}M_0V_1^2 = \frac{1}{2M_0}(M_0V_0 - \hbar k)^2 = \frac{1}{2}M_0V_0^2 - \hbar V_0.k + \frac{(\hbar k)^2}{2M_0} \quad (8)$$

then by comparing (8) and (6) it could be concluded that

$$\varepsilon_k = \hbar V_0.k - \frac{(\hbar k)^2}{2M_0} \;\rightarrow\; V_{crit} = min\left[\frac{\varepsilon_k + \frac{(\hbar k)^2}{2M_0}}{\hbar k}\right] \quad (9)\ \&\ (10)$$

And for very large M_0, V_{crit} will be obtained as follows :

$$V_{crit} = min\left[\frac{\varepsilon_k}{\hbar k}\right] \quad (11)$$

Relations (10) and (11) emphasis on finding the minimum or min of $\left[\frac{\varepsilon_k + \frac{(\hbar k)^2}{2M_0}}{\hbar k}\right]$ or $\left[\frac{\varepsilon_k}{\hbar k}\right]$. The reason for this is V_0 will be at its lowest value when $V_0.k = V_0k$, (i.e V_0 is parallel to k) and at such condition the minimum of V_0 will be the critical speed V_{crit} in which, if exceeded by the object of M_0 , M_0 will start experiencing a viscous force , thus the concept of superfluidity will be undermined for values of V_0 exceeding V_{crit} . In other words for any object moving with a speed below V_{crit} no excitations can be produced and hence the object will experience zero viscous force.

Similarly to M_0 interacting with liquid He_4 a fluid within a container also interacts with the boundaries of its container. Therefore the understanding of zero viscosity could be extended to a form that as long as the fluid moves with speeds below V_{crit} and temperatures below T_0 it will experience zero viscosity and therefore the fluid is considered a superfluid.

4-Experimental values of V_{crit}

Now that the mathematical formulation of fluid speeds where superfluidity occurs has been covered, the question is, what relation between ε_k and the wavevector k will yield the correct value of V_{crit}. Replacing $\varepsilon_k = \frac{(\hbar k)^2}{2m}$ in equation (11) will result in $V_{crit} = 0$ and $k = 0$. However it turns out that for small values of k in liquid He_4 the spectrum of excitations will take the form of $\varepsilon_k = \hbar v_s k$, where v_s represents the speed of sound. For larger values of k , ε_k has a local minimum known as the roton minimum that corresponds to a wavevector of k_0. Rotons are elementary excitations with very short wavelengths that have a value in the proximity of 3 A^o and the fact that for larger values of k , ε_k is in its local minimum (aka roton minimum) , corresponds to the minimum of $\left[\frac{\varepsilon_k}{\hbar k}\right]$ used in equation (11) . Experimentally however the obtained value of the wavevector k_0 for liquid He_4 corresponds to a minimum in dispersion, with an energy Δ. Replacing Δ and k_0 in relation (11) returns the value of 50 m/s for liquid He_4. This value is obviously less than v_s as it could be observed in figure (1)

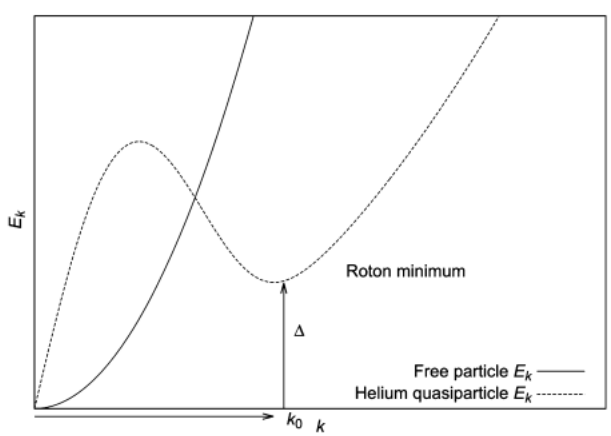

Figure (1): excitation spectrum of superfluid He(4) compared with free particle dispersion

5-What about Fermions

So far the background and theory that was covered only focused on bosons. However one may ask what happens when the particles of interest are fermionic. In 1972 David Lee showed that superfluidity also can be achieved for He_3 (He_3 is fermionic), but at temperatures much lower in comparison to the critical temperature of He_4(the critical temperature for He_3 is of the order of milli-Kelvins). $He_3$is a fermionic atom therefore the bosons that condensate are not single atoms and they are rather a pair of atoms that form at very low temperature. Such pairing of fermions into bosonic pairs also happens when electrons from copper pairs cause superconductivity at very low temperatures or is observed in condensation of ultracold Fermi gasses.

6-Conclusion:

In this report the mechanism of superfluidity has been explained in summary and even though every Bose-Einstein condensate is not a superfluid, Bose Einstein condensation does a good job at explaining the mechanism behind superfluidity. Moreover lack of viscosity was explained with the purpose of obtaining the numbers and values related to superfluidity, and to further tie the mechanism of superfluidity to Bose-Einstein Condensation an example of the condensation of a fermionic atom in form of bosonic pair of fermions was given.

Reference:

1- Kennet, M, Essential Statistical Physics 1st Edition**,** Cambridge University Press, 2021
2-Osheroff et al, Evidence for a New Phase of Solid He_3, Phys. Rev. Lett. **28**, 885 – Published 3 April 1972
3-Dr Luca Bombelli's Statistical Mechanics handwritten lecture notes, University of Mississippi, Oxford, MS, Spring semester 2022
4-Pathria. R and Beale,Paul, Statistical Mechanics 3rd Edition, Elsevier Ltd. 2011