

Physics 652: Assignment 3

(to be submitted by Thursday, March 27, 2025)

I invite you to attempt Assignment 3 and to submit all your work for Questions 1, 2, and 4. Please turn in a paper copy in class; and email to kbeach@olemiss.edu a single Wolfram Notebook, sent as an attachment, that contains the relevant Mathematica code. Use the naming convention Phys652-A3-webid.nb, and be sure to include Phys652-Spring2025-webid Assignment 3 Submission on the subject line.

1. The trajectory $x(t)$ obeys the differential equation

$$\ddot{x} + \alpha \dot{x} + 2e^{-\alpha t} x = 0,$$

where $\alpha = 1/10$, subject to the initial conditions $x(0) = 1$, $\dot{x}(0) = -1$. For this system, *Mathematica* is able to generate both numerical and analytical solutions:

```
sol1 = NDSolve[{x''[t] + (1/10) x'[t] + 2 Exp[-t/10] x[t] == 0,
  x[0] == 1, x'[0] == -1}, x, {t, 0, 100}]
Plot[Evaluate[x[t] /. sol1], {t, 0, 100}, PlotRange -> All]
sol2 = DSolve[{x''[t] + (1/10) x'[t] + 2 Exp[-t/10] x[t] == 0,
  x[0] == 1, x'[0] == -1}, x, t]
Plot[x[t] /. sol2, {t, 0, 100}, PlotRange -> All]
```

Solve for $x(t)$ by assuming a series solution of the form

$$x(t) = t^r \sum_{n=0}^{\infty} a_n t^n.$$

You should be able to show that

$$x(t) = 1 - t - \frac{19}{20}t^2 + \frac{239}{600}t^3 + O(t^4).$$

Here is a quick check that those coefficients are correct:

```
N[Coefficient[Series[First[x[t] /. sol2], {t, 0, 5}], t, {1, 2, 3}]]
N[{-1, -19/20, 239/600}]
```

At this level of approximation, deviations from the exact solution become significant around $t \approx 1$:

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Plot[{x[t] /. sol2, 1 - t - (19/20) t^2 + (239/600) t^3}, {t, 0, 2}, PlotRange -> All]
```

2. A gas of molecules is held in a closed container. At any given moment, some portion of the gas, of mass m , has deposited on (and adhered to) the interior surface of the container. We refer to this as the adsorbate. The remaining mass μ remains in vapor form. Mass sublimates from the surface at a uniform rate α , and there is a flux of molecules returning to the surface, denoted by β . Finally, the chamber is being evacuated by a pump at a rate γ . This picture leads to a coupled pair of rate equations:

$$\begin{aligned} \dot{m} &= -\alpha m + \beta \mu \\ \dot{\mu} &= +\alpha m - (\beta + \gamma)\mu \end{aligned}$$

- (a) Determine the general time-dependent solutions $m(t)$ and $\mu(t)$ in terms of α , β , γ and the initial conditions $m(0) = m_0$ and $\mu(0) = \mu_0$.
- (b) Show that, for fast pumping $\gamma \gg \alpha, \beta$ and times $t \gg (\gamma + \beta + \alpha\beta/\gamma)^{-1}$, the mass deposited on the container walls decays according to

$$m(t) = m_0 \left(1 + \frac{\beta - \alpha}{\gamma} + \frac{\alpha\beta}{\gamma^2} \right) e^{-\alpha(1-\beta/\gamma)t}.$$

3. The function $y(x)$ is the solution to the second-order differential equation

$$y'' + 3x^2y + x^3y' = 1.$$

Use the exactness property to show that

$$y(x) = e^{-x^4/4} \left[y(0) + \int_0^x d\xi (y'(0) + \xi) e^{\xi^4/4} \right].$$

4. Solve the eigenvalue problem $L_0 y = \lambda y$ for the function $y(x)$ defined on the interval $[0, 1]$ with boundary conditions $y(0) = y(1) = 0$. The differential operator is

$$L_0 = \frac{d^2}{dx^2} + 2 \frac{d}{dx} + 2.$$

- (a) Separately examine the cases $\lambda < 1$, $\lambda = 1$, and $\lambda > 1$. Be sure to report the form of the $y(x)$ eigenfunctions and the allowed values of the λ eigenvalues.
- (b) Put the problem into Sturm-Liouville form, $Lu = \lambda \rho u$, with a weight function $\rho(x) = e^{2x}$. (Be aware that $L \neq L_0$.) Show that the set of eigenfunctions $\{u_n(x) : n = 1, 2, 3, \dots\}$ is orthonormal with respect to the inner product $(a, b) = \int_0^1 \rho(x) dx a(x)b(x)$.
- (c) Have a look at the Green's function

$$G(x, x') = \sum_{n=1}^{\infty} \frac{u_n(x)u_n(x')}{\lambda_n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{-x-x'} \sin(n\pi x) \sin(n\pi x')}{1 - n^2\pi^2}.$$

```
G[x_, y_] = (1/2) Sum[Exp[-x - y] Sin[n \[Pi] x] Sin[n \[Pi] y] / (1 + n^2 \[Pi]^2),
  {n, 1, 10000}];
DensityPlot[G[x, y], {x, 0, 1}, {y, 0, 1}]
```

Prove that it's is a solution to the differential equation $LG(x, x') = \delta(x - x')$.