Physics 652: Assignment 5 (optional bonus)

(to be submitted by Thursday, May 2, 2024)

1. The Bessel function of the first kind of order ν and of real argument x has a series expansion

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)} \left(\frac{x}{2}\right)^{2k+\nu}$$

   (a) Compute the Wronskian of $J_\nu(x)$ and $J_{-\nu}(x)$.

   (b) Argue that $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent if ν is not an integer.

2. The zeros $z_{n,k}$ of the nth Bessel function $J_n(x)$ are infinite in number. They are labelled $k = 1, 2, 3, \ldots$ and satisfy $J_n(z_{n,k}) = 0$. We can access them in *Mathematica* as follows.

```
N[BesselJZero[0, 1]]
BesselJ[0, BesselJZero[0, 1]]
Plot[BesselJ[1, z], {z, 0, 15},
 Epilog -> {PointSize[0.03], Red,
  Point[Table[{BesselJZero[1, k], 0}, {k, 4}]]}]
```

   (a) For large n and k, the zeros $z_{n,k}$ approach the limiting value $k\pi + (n+1)\pi/2 + \pi/4$. Consider the zeros of J_0, J_1, J_2, and J_3. Plot the differences between successive zeros $(z_{n,k+1} - z_{n,k})$ and show that they converge to the asymptotic value π.

   (b) The function $f(x)$ is given by $f(x) = (1+x)(2-x)$ for all $x \in [0, 2]$. Reproduce the function on that interval by computing the Fourier-Bessel series

$$f(x) = x^\alpha \sum_{k=1}^{\infty} a_k J_n(z_{n,k} x/2)$$

   with expansion coefficients

$$a_k = \frac{1}{2[J_{n+1}(z_{n,k})]^2} \int_0^2 dy\, y^{1-\alpha} J_n(z_{n,k} y/2) f(y).$$

   You are free to choose a value of $0 \le \alpha \le 1$

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f[x_] = (1+x)(2-x)
c[n_,k_] := 1/(2 BesselJ[n+1,BesselJZero[n,k]]^2)
Integrate[y^(1-\[Alpha])f[y] BesselJ[n,BesselJZero[n,k] y/2],{y,0,2}]
```