Physics 652: Assignment 3

(to be submitted by Tuesday, March 26, 2024)

1. The trajectory $x(t)$ obeys the differential equation

$$\ddot{x} + \alpha\dot{x} + 2e^{-\alpha t}x = 0,$$

where $\alpha = 1/10$, subject to the initial conditions $x(0) = 1$, $\dot{x}(0) = -1$. For this system, *Mathematica* is able to generate both numerical and analytical solutions:

```
sol1 = NDSolve[{x''[t] + (1/10) x'[t] + 2 Exp[-t/10] x[t] == 0,
  x[0] == 1, x'[0] == -1}, x, {t, 0, 100}]
Plot[Evaluate[x[t] /. sol1], {t, 0, 100}, PlotRange -> All]
sol2 = DSolve[{x''[t] + (1/10) x'[t] + 2 Exp[-t/10] x[t] == 0,
  x[0] == 1, x'[0] == -1}, x, t]
Plot[x[t] /. sol2, {t, 0, 100}, PlotRange -> All]
```

Solve for $x(t)$ by assuming a series solution of the form

$$x(t) = t^r \sum_{n=0}^{\infty} a_n t^n.$$

You should be able to show that

$$x(t) = 1 - t - \frac{19}{20}t^2 + \frac{239}{600}t^3 + O(t^4).$$

Here is a quick check that those coefficients are correct:

```
N[Coefficient[Series[First[x[t] /. sol2], {t, 0, 5}], t, {1, 2, 3}]]
N[{-1, -19/20, 239/600}]
```

At this level of approximation, deviations from the exact solution become significant around $t \approx 1$:

```
Plot[{x[t] /. sol2, 1 - t - (19/20) t^2 + (239/600) t^3}, {t, 0, 2}, PlotRange -> All]
```

2. A gas of molecules is held in a closed container. At any given moment, some portion of the gas, of mass m, has deposited on (and adhered to) the interior surface of the container. We refer to this as the adsorbate. The remaining mass μ remains in vapor form. Mass sublimates from the surface at a uniform rate α, and there is a flux of molecules returning to the surface, denoted by β. Finally, the chamber is being evacuated by a pump at a rate γ. This picture leads to a coupled pair of rate equations:

$$\dot{m} = -\alpha m + \beta\mu$$
$$\dot{\mu} = +\alpha m - (\beta + \gamma)\mu$$

   (a) Determine the general time-dependent solutions $m(t)$ and $\mu(t)$ in terms of α, β, γ and the initial conditions $m(0) = m_0$ and $\mu(0) = \mu_0$.

   (b) Show that, for fast pumping $\gamma \gg \alpha, \beta$ and times $t \gg (\gamma + \beta + \alpha\beta/\gamma)^{-1}$, the mass deposited on the container walls decays according to

$$m(t) = m_0\left(1 + \frac{\beta - \alpha}{\gamma} + \frac{\alpha\beta}{\gamma^2}\right)e^{-\alpha(1-\beta/\gamma)t}.$$

3. The function $y(x)$ is the solution to the second-order differential equation

$$y'' + 3x^2 y + x^3 y' = 1.$$

Use the exactness property to show that

$$y(x) = e^{-x^4/4}\left[y(0) + \int_0^x d\xi \left(y'(0) + \xi\right) e^{\xi^4/4}\right].$$

4. Solve the eigenvalue problem $L_0 y = \lambda y$ for the function $y(x)$ defined on the interval $[0, 1]$ with boundary conditions $y(0) = y(1) = 0$. The differential operator is

$$L_0 = \frac{d^2}{dx^2} + 2\frac{d}{dx} + 2.$$

(a) Separately examine the cases $\lambda < 1$, $\lambda = 1$, and $\lambda > 1$. Be sure to report the form of the $y(x)$ eigenfunctions and the allowed values of the λ eigenvalues.

(b) Put the problem into Sturm-Liouville form, $Lu = \lambda \rho u$, with a weight function $\rho(x) = e^{2x}$. (Be aware that $L \neq L_0$.) Show that the set of eigenfunctions $\{u_n(x) : n = 1, 2, 3, \ldots\}$ is orthonormal with respect to the inner product $(a, b) = \int_0^1 \rho(x)\, dx\, a(x) b(x)$.

(c) Have a look at the Green's function

$$G(x, x') = \sum_{n=1}^{\infty} \frac{u_n(x) u_n(x')}{\lambda_n} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{-x-x'} \sin(n\pi x) \sin(n\pi x')}{1 - n^2\pi^2}.$$

```
G[x_, y_] = (1/2) Sum[Exp[-x - y] Sin[n \[Pi] x] Sin[n \[Pi] y] / (1 + n^2
    \[Pi]^2), {n, 1, 10000}];
DensityPlot[G[x, y], {x, 0, 1}, {y, 0, 1}]
```

Prove that it's is a solution to the differential equation $LG(x, x') = \delta(x - x')$.