Physics 652: Assignment 2

(to be submitted by Thursday, February 22, 2024)

1. A function $F(x)$ has slope $u = dF/dx$. The Legendre transformation gives us a new function $G(u) = F - ux$ that is an explicit function of the slope.

   (a) Explain what f is in the expression $G(u) = F(f^{-1}(u)) - uf^{-1}(u)$.

   (b) Use *Mathematica* to make a nice plot of the function $F(x) = -x^3 + x^2 + 2x$ between -1 and 2. Then use this code snippet

```
F[x_] = -x^3 + x^2 + 2 x
f = F'
Solve[f[x] == u, x]
{sol1, sol2} = Solve[f[x] == u, x]
G1[u_] = Simplify[F[x /. sol1] - u (x /. sol1)]
Plot[{G1[u], G2[u]}, {u, -2, 7/3}]
```

   to construct and plot functions of the form $G(u) = F(x) - ux$. Why are there two? Explain what's going on in the plot of $G(u)$ versus u and what the relation is to the plot you made of $F(x)$ versus x.

2. Consider a system in coordinates q and $\dot{q}$, described by a Lagrangian

$$L(q, \dot{q}) = \frac{m}{2!}\dot{q}^2 + \frac{\kappa}{4!}\dot{q}^4 - \frac{m\omega_0^2}{2!}q^2 - \frac{\lambda}{4!}q^4.$$

   (a) Derive the dynamical equations prescribed by Euler-Lagrange. You should be able to show that

$$\ddot{q} = -\omega_0^2 q\left(\frac{1 - (\lambda/6m\omega_0^2)q^2}{1 + (\kappa/2m)\dot{q}^2}\right).$$

   (b) Apply the Ansatz $q(t) = A\cos\omega_0 t + \delta q(t)$. Derive the leading order expression for $\delta\ddot{q}$.

   (c) Compute the conjugate momentum $p \equiv \partial L/\partial\dot{q}$.

   (d) The corresponding Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{\kappa p^4}{24m^4} + \frac{\kappa^2 p^6}{72m^7} - \frac{\kappa p^8}{144m^{10}} + O(p^{10}) + \frac{m\omega_0^2 q^2}{2} + \frac{\lambda q^4}{24}.$$

   Verify by hand the terms up to $O(p^4, q^4)$. The computer can do it for you to arbitrary order:

```
L[qq_, q_] = (m/2!)qq^2 + (\[Kappa]/4!)qq^4 - (m\[Omega]^2/2!)q^2
  - (\[Lambda]/4!)q^4
pdef = D[L[qq, q], qq]
{sol1, sol2, sol3} = Solve[p == pdef, qq]
qq1=qq/.sol1
Assuming[{m > 0, \[Kappa] > 0}, Series[qq /. sol1, {p, 0, 9}]]
Map[Simplify, Assuming[{m > 0, \[Kappa] > 0}, Series[p qq1 - L[qq1, q], {p, 0, 9}]]]
```

3. The function $x(t)$ is a solution to the third-order, constant-coefficient differential equation $\dddot{x} - 2\ddot{x} + \dot{x} = 0$.

   (a) Solve it with Mathematica.

   ```
   DSolve[D[x[t], {t, 3}] - 2 D[x[t], {t, 2}] + D[x[t], t] == 0, x[t], t]
   ```

   (b) Here is a solution strategy that you can carry out by hand. To start, integrate the differential equation once to arrive at

   $$\ddot{x} - 2\dot{x} + x = \text{constant} \equiv c_0.$$

   Then substitute $x = c_0 + t^a e^{zt}$ into $\ddot{x} - 2\dot{x} + x - c_0 = 0$. Show that

   $$0 = \left\{[(a+zt)^2 - a] - 2(a+zt)t + t^2\right\}t^{a-2}e^{zt}.$$

   (c) Consider appropriate values of a and z. Think about which values will solve the equation in part (b) and how many independent solutions you need. Write an expression for the most general $x(t)$.

   (d) Prove that the difference $x^{(n+1)}(0) - x^{(n)}(0)$ is independent of n and that $ex^{(n+1)}(0) - x^{(n)}(1)\ \forall n$.