Physics 652: Assignment 1

(to be submitted by Thursday, February 8, 2024)

1. For each the following, identify whether the differential equation is ordinary or partial, linear or nonlinear. If it is linear, indicate whether we should categorize it as homogeneous or inhomogenous.

$$\text{(i)} \quad \partial_y \phi + \partial_x^3 \phi - 6\,\phi\,\partial_x \phi = 0$$

$$\text{(ii)} \quad x f''(x) - (1 - x^2) f'(x) + x^{-1} f(x) + \frac{1 - x^2}{1 + x^2} = 0$$

$$\text{(iii)} \quad i\partial_t \psi = -\frac{1}{2}\partial_x^2 \psi + \kappa |\psi|^2 \psi$$

$$\text{(iv)} \quad \left(x \frac{d^2}{dx^2} - x^2 \frac{d}{dx} \right) g(x) = 0$$

2. Use this *Mathematica* command

```
DSolve[x'[t] == a x[t] (1 - x[t]/X), x[t], t]
```

to obtain the solution to the *logistic equation*,

$$\dot{x} = \frac{dx}{dt} = ax\left(1 - \frac{x}{X}\right).$$

Try out this next code snippet to check that the purported solution actually solves the logistic equation:

```
rhslogisticeqn = a x[t] (1 - x[t]/X)
soln = First[DSolve[x'[t]==rhslogisticeqn, x[t], t]]
Simplify[D[x[t]/.soln,t]==rhslogisticeqn/.soln]
```

Now show explicitly (by hand) that

$$x(t) = \frac{e^{at+bX} X}{e^{at+bX} - 1} = \frac{X x_0 e^{at}}{X + x_0 (e^{at} - 1)}$$

[with $x_0 = x(0)$] is a solution to the ODE.

3. Separate the variables of $(1 + y^2) y\, dx + (1 + x^2) x\, dy = 0$. Find its general integral and solution $y(x)$.

4. Determine whether

$$(1 + x^2 + y^2)^{-3/2} \left[(1 + y^2) y\, dx + (1 + x^2) x\, dy \right] = 0$$

is exact (using the $P_y = Q_x$ test). Find its general integral and solution $y(x)$. Explain why you can also solve the equation with this *Mathematica* command:

```
DSolve[y'[x] == -(1 + y[x]^2) y[x]/((1 + x^2) x), y[x], x]
```

5. Obtain the general solution to the differential equation $y' + y/x = c/x$ with machine assistance. Then try to arrive at the solution by hand.

```
DSolve[y'[x] + y[x]/x == c/x, y[x], x]
```

6. Obtain the general solution to the differential equation $y' + xy = ce^{-x^2/2}$ with machine assistance. Then try to arrive at the solution by hand.

```
DSolve[y'[x] + x y[x] == c Exp[-x^2/2], y[x], x]
```

7. Consider three potentials (Coulombic/gravitational) $V_1(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$, (harmonic) $V_2(x, y, z) = x^2 + y^2 + z^2$, and (mishegoss) $V_3(x, y, z) = x^4 + 2x^3y^3/z^2 + y^4 + 2y^3z^3/x^2 + z^4$.

   (a) Show that each of V_1, V_2, and V_3 is a homogenous function of its arguments. In each case, determine the degree of homogeneity, n.

   (b) Establish that the Euler theorem

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = nV$$

   holds.

   (c) Suppose that a particle of mass m with kinetic energy $K = (m/2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ is allowed to come into thermal equilibrium at temperature T. Equipartition suggests that the average kinetic energy is $\langle K \rangle = (3/2)k_B T$. What are the average potential energies $\langle V_i \rangle$ for each of $i = 1, 2, 3$.

Here's a way to automate the calculation for parts (a) and (b):

```
V1[x_, y_, z_] = 1/Sqrt[x^2 + y^2 + z^2]
n1 = Refine[Log[Refine[Simplify[Refine[
     V1[t x, t y, t z]/V1[x, y, z],
     {x \[Element] Reals, y \[Element] Reals, z \[Element] Reals}]],
       t > 0]]/Log[t], t > 0]
Simplify[
   x D[V1[x, y, z], x] + y D[V1[x, y, z], y] + z D[V1[x, y, z], z] == n1 V1[x, y, z]]
```