Physics 652: Assignment 2

(to be submitted by Thursday, February 16, 2023)

1. Consider three potentials (Coulombic/gravitational) $V_1(x, y, z) = 1/\sqrt{x^2 + y^2 + z^2}$, (harmonic) $V_2(x, y, z) = x^2 + y^2 + z^2$, and (mishegoss) $V_3(x, y, z) = x^4 + 2x^3y^3/z^2 + y^4 + 2y^3z^3/x^2 + z^4$.

   (a) Show that each of V_1, V_2, and V_3 is a homogenous function of its arguments. In each case, determine the degree of homogeneity, n.

   (b) Establish that the Euler theorem

   $$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = nV$$

   holds.

   (c) Suppose that a particle of mass m with kinetic energy $K = (m/2)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ is allowed to come into thermal equilibrium at temperature T. Equipartition suggests that the average kinetic energy is $\langle K \rangle = (3/2)k_B T$. What are the average potential energies $\langle V_i \rangle$ for each of $i = 1, 2, 3$.

   Here's a way to automate the calculation for parts (a) and (b):

```
V1[x_, y_, z_] = 1/Sqrt[x^2 + y^2 + z^2]
n1 = Refine[Log[Refine[Simplify[Refine[
     V1[t x, t y, t z]/V1[x, y, z]],
     {x \[Element] Reals, y \[Element] Reals, z \[Element] Reals}]],
       t > 0]]/Log[t], t > 0]
Simplify[
   x D[V1[x, y, z], x] + y D[V1[x, y, z], y] + z D[V1[x, y, z], z] == n1 V1[x, y, z]]
```

2. A function $F(x)$ has slope $u = dF/dx$. The purpose of the Legendre transformation is to construct a new function $G(u) = F - ux$ that is an explicit function of the slope.

   (a) Explain what f is in the expression $G(u) = F(f^{-1}(u)) - uf^{-1}(u)$.

   (b) Use *Mathematica* to make a nice plot of the function $F(x) = -x^3 + x^2 + 2x$ between -1 and 2. Then use this code snippet

```
F[x_] = -x^3 + x^2 + 2 x
f = F'
Solve[f[x] == u, x]
{sol1, sol2} = Solve[f[x] == u, x]
G1[u_] = Simplify[F[x /. sol1] - u (x /. sol1)]
Plot[{G1[u], G2[u]}, {u, -2, 7/3}]
```

   to construct and plot functions of the form $G(u) = F(x) - ux$. Why are there two? Explain what's going on in the plot of $G(u)$ versus u and what the relation is to the plot you made of $F(x)$ versus x.

3. Consider a system in coordinates q and $\dot{q}$, described by a Lagrangian

   $$L(q, \dot{q}) = \frac{m}{2!}\dot{q}^2 + \frac{\kappa}{4!}\dot{q}^4 - \frac{m\omega_0^2}{2!}q^2 - \frac{\lambda}{4!}q^4.$$

   (a) Derive the dynamical equations prescribed by Euler-Lagrange. You should be able to show that

   $$\ddot{q} = -\omega_0^2 q\left(\frac{1 - (\lambda/6m\omega_0^2)q^2}{1 + (\kappa/2m)\dot{q}^2}\right).$$

(b) Apply the Ansatz $q(t) = A\cos\omega_0 t + \delta q(t)$. Derive the leading order expression for $\delta\ddot{q}$.

(c) Compute the conjugate momentum $p \equiv \partial L/\partial \dot{q}$.

(d) The corresponding Hamiltonian is

$$H = \frac{p^2}{2m} - \frac{\kappa p^4}{24m^4} + \frac{\kappa^2 p^6}{72m^7} - \frac{\kappa p^8}{144m^{10}} + O(p^{10}) + \frac{m\omega_0^2 q^2}{2} + \frac{\lambda q^4}{24}.$$

Verify by hand the terms up to $O(p^4, q^4)$. The computer can do it for you to arbitrary order:

```
L[qq_, q_] = (m/2!)qq^2 + (\[Kappa]/4!)qq^4 - (m\[Omega]^2/2!)q^2 - (\[Lambda]/4!)q^4
pdef = D[L[qq, q], qq]
{sol1, sol2, sol3} = Solve[p == pdef, qq]
qq1=qq/.sol1
Assuming[{m > 0, \[Kappa] > 0}, Series[qq /. sol1, {p, 0, 9}]]
Map[Simplify, Assuming[{m > 0, \[Kappa] > 0}, Series[p qq1 - L[qq1, q], {p, 0, 9}]]]
```