Phys 726 – Assignment 6 Solutions

1(a) $\hat{S}^z = \frac{1}{2} \sum_{\alpha\beta} c_\alpha^\dagger \sigma^z c_\beta = \frac{1}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)$

$$\hat{S}^z |0\rangle = \frac{1}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)|vac\rangle = 0 = 0 \cdot |0\rangle$$

$$\hat{S}^z |\uparrow\rangle = \frac{1}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow) c_\uparrow^\dagger |vac\rangle$$

$$= \frac{1}{2}\left(c_\uparrow^\dagger c_\uparrow c_\uparrow^\dagger - c_\uparrow^\dagger \cancel{c_\downarrow^\dagger c_\downarrow}^{\to 0}\right)|vac\rangle$$

$$= \frac{1}{2} c_\uparrow^\dagger \left(1 - \cancel{c_\uparrow^\dagger c_\uparrow}^{\to 0}\right)|vac\rangle$$

$$= \frac{1}{2} c_\uparrow^\dagger |vac\rangle = \frac{1}{2}|\uparrow\rangle$$

$$\hat{S}^z |\downarrow\rangle = \frac{1}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow) c_\downarrow^\dagger |vac\rangle$$

$$= \frac{1}{2}\left(c_\downarrow^\dagger \cancel{c_\uparrow^\dagger c_\uparrow}^{\to 0} - c_\downarrow^\dagger c_\downarrow c_\downarrow^\dagger\right)|vac\rangle$$

$$= -\frac{1}{2} c_\downarrow^\dagger \left(1 - \cancel{c_\downarrow^\dagger c_\downarrow}^{\to 0}\right)|vac\rangle$$

$$= -\frac{1}{2}|\downarrow\rangle$$

$$\hat{S}^z|\uparrow\downarrow\rangle = \frac{1}{2}(\hat{n}_\uparrow - \hat{n}_\downarrow)c_\uparrow^\dagger c_\downarrow^\dagger|vac\rangle$$

$$= \frac{1}{2}(c_\uparrow^\dagger c_\uparrow c_\uparrow^\dagger c_\downarrow^\dagger - c_\uparrow^\dagger c_\downarrow^\dagger c_\downarrow c_\downarrow^\dagger)|vac\rangle$$

$$= \frac{1}{2}(c_\uparrow^\dagger(1-\cancel{c_\uparrow^\dagger c_\uparrow}^{\to 0})c_\downarrow^\dagger - c_\uparrow^\dagger c_\downarrow^\dagger(1-\cancel{c_\downarrow^\dagger c_\downarrow}^{\to 0}))|vac\rangle$$

$$= \frac{1}{2}(c_\uparrow^\dagger c_\downarrow^\dagger - c_\uparrow^\dagger c_\downarrow^\dagger)|vac\rangle = 0$$

$$= 0 \cdot |\uparrow\downarrow\rangle$$

(b) $$\hat{S}^2 = \hat{\vec{S}} \cdot \hat{\vec{S}} = \frac{1}{2}\sum_{\alpha\beta} c_\alpha^\dagger \vec{\sigma}_{\alpha\beta} c_\beta \cdot \frac{1}{2}\sum_{\mu\nu} c_\mu^\dagger \sigma_{\mu\nu} c_\nu$$

$$= \frac{1}{4}\sum_{\substack{\alpha\beta\\ \mu\nu}} \underbrace{\vec{\sigma}_{\alpha\beta}\cdot\vec{\sigma}_{\mu\nu}}_{= 2\delta_{\alpha\nu}\delta_{\beta\mu} - \delta_{\alpha\beta}\delta_{\mu\nu}} c_\alpha^\dagger c_\beta c_\mu^\dagger c_\nu$$

$$= \frac{1}{4}\left[2\sum_{\alpha\beta} c_\alpha^\dagger c_\beta c_\beta^\dagger c_\alpha - \sum_{\alpha\beta} c_\alpha^\dagger c_\alpha c_\beta^\dagger c_\beta\right]$$

$$= \frac{1}{4}\left[2\sum_{\alpha\beta} c_\alpha^\dagger c_\beta(\delta_{\alpha\beta} - c_\alpha c_\beta^\dagger) - \left(\sum_\alpha c_\alpha^\dagger c_\alpha\right)\left(\sum_\beta c_\beta^\dagger c_\beta\right)\right]$$

$$= \frac{1}{4}\left[2\sum_\alpha c_\alpha^\dagger c_\alpha + 2\sum_{\alpha\beta} c_\alpha^\dagger c_\alpha c_\beta c_\beta^\dagger - \left(\sum_\alpha c_\alpha^\dagger c_\alpha\right)\left(\sum_\beta c_\beta^\dagger c_\beta\right)\right]$$

$$= \frac{1}{4}\left[2\hat{n} + 2\sum_\alpha c_\alpha^\dagger c_\alpha \sum_\beta (1 - c_\beta^\dagger c_\beta) - \hat{n}^2\right]$$

$$= \frac{1}{4}\left[2\hat{n} + 2\hat{n}(2-\hat{n}) - \hat{n}^2\right]$$

$$= \frac{3}{4}\hat{n}(2-\hat{n})$$

(c) An SU(2) quantum spin has states $|\uparrow\rangle, |\downarrow\rangle$ that are eigenstates of $\hat{S}^z$ with $\hat{S}^z|\uparrow\rangle = \frac{1}{2}|\uparrow\rangle$ and $\hat{S}^z|\downarrow\rangle = \frac{1}{2}|\downarrow\rangle$, and it has an overall spin magnitude $\hat{S}^2|\uparrow\rangle = \frac{1}{2}(\frac{1}{2}+1)|\uparrow\rangle$

$$= \frac{3}{4}|\uparrow\rangle$$

$$\hat{S}^2|\downarrow\rangle = \frac{3}{4}|\downarrow\rangle$$

A fermion in a localized energy ~~band~~ level, subject to the restriction $\hat{n}=1$, behaves in exactly the same way.

2(a)

Coefficients $C_S^{(N)}$

N	0	1/2	1	3/2	2	5/2	3	7/2	4	9/2	total
1		1									2
2	1		1								4
3		2		1							8
4	2		3		1						16
5		5		4		1					32
6	5		9		5		1				64
7		14		14		6		1			128
8	14		28		20		7		1		256
9		42		48		27		8		1	512

maximal spin value at $S=\frac{N}{2}$

N even

N odd

$$C_0^{(N)} = \frac{1}{\frac{N}{2}+1}\binom{N}{N/2} = \frac{N!}{\left(\frac{N}{2}\right)!\left(\frac{N}{2}+1\right)!}$$

$$C_{1/2}^{(N)} = \frac{1}{\frac{N+1}{2}+1}\binom{N+1}{\frac{N+1}{2}} = \frac{(N+1)!}{\left(\frac{N+1}{2}\right)!\left(\frac{N+3}{2}\right)!}$$

$$\sum_{S=0}^{N/2}(2S+1) = 2^N$$

or $$\sum_{S=1/2}^{N/2}(2S+1) = 2^N$$

(b) $\vec{S}^{tot} = \sum_i \vec{S}_i$ 5

$$(\hat{\vec{S}}^{tot})^2 = \sum_{ij} \vec{S}_i \cdot \vec{S}_j = \left(\sum_{i=j} + \sum_{i<j} + \sum_{i>j} \right) \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j$$

$$= \sum_i \hat{S}_i^2 + 2 \sum_{i<j} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j$$

$$= \frac{3N}{4} + 2 \sum_{i<j} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j$$

$$\Rightarrow \quad \hat{H} = -2|J| \sum_{i<j} \hat{\vec{S}}_i \cdot \hat{\vec{S}}_j = |J| \left(\frac{3N}{4} - (\hat{S}^{tot})^2 \right)$$

The ground state is the maximally aligned state with total spin $S^{tot} = \frac{N}{2}$,

energy $E = |J| \left(\frac{3N}{4} - S^{tot}(S^{tot}+1) \right)$

$$= |J| \left(\frac{3N}{4} - \frac{N}{2}\left(\frac{N}{2}+1\right) \right)$$

$$= \frac{|J|}{4} \left(3N - N(N+2) \right)$$

$$= \frac{|J|}{4} (N - N^2) = \frac{|J|}{4} N(1-N),$$

and ground state degeneracy

$$2S^{tot}+1 = N+1$$

3(a) Quantum Heisenberg model

$$\hat{H} = \frac{1}{2}\sum_{jj'} J_{jj'} \sum_{a=x,y,z} \hat{S}^a_j \hat{S}^a_{j'}$$

$$[\hat{H}, \hat{S}^b_\ell] = \frac{1}{2}\sum_{jj'} J_{jj'} \sum_a \underbrace{[\hat{S}^a_j \hat{S}^a_{j'}, \hat{S}^b_\ell]}$$

$$\hat{S}^a_j [\hat{S}^a_{j'}, \hat{S}^b_\ell] + [\hat{S}^a_j, \hat{S}^b_\ell]\hat{S}^a_{j'}$$

$$= \hat{S}^a_j \delta_{j'\ell}\, i \sum_c \varepsilon^{abc} \hat{S}^c_\ell$$

$$+ \delta_{j\ell}\, i \sum_c \varepsilon^{abc} \hat{S}^c_\ell \hat{S}^a_{j'}$$

$$= i \sum_c \varepsilon^{abc} \left(\delta_{j'\ell} \hat{S}^a_j \hat{S}^c_\ell + \delta_{j\ell} \hat{S}^c_\ell \hat{S}^a_{j'} \right)$$

$$[\hat{H}, \hat{S}^b_\ell] = \frac{1}{2} i \sum_{a,c} \varepsilon^{abc} \sum_j \left(J_{j\ell} \hat{S}^a_j \hat{S}^c_\ell + J_{\ell j} \hat{S}^c_\ell \hat{S}^a_j \right)$$

$$= i \sum_{a,c} \varepsilon^{abc} \sum_{j\neq \ell} J_{j\ell} \hat{S}^a_j \hat{S}^c_\ell \quad \text{since } J_{j\ell} = J_{\ell j} \sim (1-\delta_{j\ell})$$

Relabelling $(a \leftrightarrow b)$ gives

$$[\hat{H}, \hat{S}_\ell^a] = i \sum_{b,c} \varepsilon^{bac} \sum_{j \neq \ell} J_{j\ell} \hat{S}_j^b \hat{S}_\ell^c$$

$$= -i \sum_{b,c} \varepsilon^{abc} \sum_{j \neq \ell} J_{j\ell} \hat{S}_j^b \hat{S}_\ell^c$$

and hence

$$\frac{d}{dt} \langle \hat{\vec{S}}_\ell \rangle = i \langle [\hat{H}, \hat{\vec{S}}_\ell] \rangle$$
$$= \sum_{j \neq \ell} J_{j\ell} \langle \hat{\vec{S}}_j \times \hat{\vec{S}}_\ell \rangle$$

(b) In a nearest-neighbour model, there are contributions of strength J when ~~$\ell = j+1$ and $\ell = j-1$~~ $j = \ell - 1$ and $j = \ell + 1$.

$$\frac{d}{dt} \langle \hat{\vec{S}}_\ell \rangle = J \langle \hat{\vec{S}}_{\ell-1} \times \hat{\vec{S}}_\ell + \hat{\vec{S}}_{\ell+1} \times \hat{\vec{S}}_\ell \rangle$$

$$= J \langle (\hat{\vec{S}}_{\ell-1} + \hat{\vec{S}}_{\ell+1}) \times \vec{S}_\ell \rangle$$

spin at site ℓ couples to the net spin of its two neighbours

(c) $$\frac{d}{dt}\vec{\Omega}_j = JS\left(\vec{\Omega}_{j-1} + \vec{\Omega}_{j+1}\right) \times \vec{\Omega}_j$$

Let $J = -|J|$

and $$\Omega_j = \sqrt{1-\alpha_j^2-\beta_j^2}\,\vec{e}_z + \alpha_j \vec{e}_x + \beta_j \vec{e}_y$$

Then

$$\frac{d}{dt}\Omega_j = \frac{1}{2}\frac{1}{\sqrt{1-\alpha_j^2-\beta_j^2}}\left(-2\alpha_j\dot{\alpha}_j - 2\beta_j\dot{\beta}_j\right)\vec{e}_z + \dot{\alpha}_j \vec{e}_x + \dot{\beta}_j \vec{e}_y$$

$$\sim \dot{\alpha}_j \vec{e}_x + \dot{\beta}_j \vec{e}_y$$ to linear order

and

$$\dot{\alpha}_j = -|J|S\left\{ \left(\Omega^y_{j-1} + \Omega^y_{j+1}\right)\Omega^z_j - \left(\Omega^z_{j-1} + \Omega^z_{j+1}\right)\Omega^y_j \right.$$

$$\simeq -|J|S\left\{ \left(\beta_{j-1} + \beta_{j+1}\right)(1) - (1+1)\beta_j \right\}$$

$$= -|J|S\left\{ \beta_{j+1} + \beta_{j+1} - 2\beta_j \right\}$$

and

$$\dot{\beta}_j = -|J|S\left\{ \left(\Omega^x_{j-1} + \Omega^x_{j+1}\right)\Omega^z_j - \left(\Omega^z_{j-1} + \Omega^z_{j+1}\right)\Omega^x_j \right\}$$

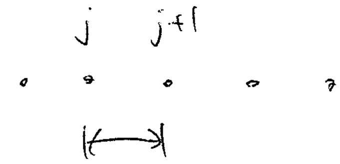
$$\simeq -|J|S\left\{ \left(\alpha_{j-1} + \alpha_{j+1}\right)(1) - (1+1)\alpha_j \right\}$$

$$= -|J|S\left\{ \alpha_{j-1} + \alpha_{j+1} - 2\alpha_j \right\}$$

(d) Consider modes with wave vector q and angular frequency ε_q

$$\alpha_j(t) = \bar{\alpha}\, e^{i(qaj - \varepsilon_q t)}$$

$$\beta_j(t) = \bar{\beta}\, e^{i(qaj - \varepsilon_q t)}$$



lattice spacing for the linear chain

$$\dot{\alpha}_j = -i\varepsilon_q \alpha_j(t) = -|J|S\Big\{ \bar{\beta} e^{i(qa(j+1) - \varepsilon_q t)} + \bar{\beta} e^{i(qa(j-1) - \varepsilon_q t)} - 2\bar{\beta} e^{i(qa - \varepsilon_q t)} \Big\}$$

$$= -|J|S\left\{ e^{iqa} + e^{-iqa} - 2 \right\} \beta(t)$$

$$= -2|J|S\,(\cos qa - 1)\,\beta(t)$$

$$\dot{\beta}_j = -i\varepsilon_q \beta_j(t) = +2|J|S\,(\cos qa - 1)\,\alpha(t)$$

i.e.
$$\frac{d}{dt}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = 2|J|S \begin{pmatrix} 0 & 1-\cos qa \\ \cos qa - 1 & 0 \end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = i\varepsilon_q \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$$

In other words,

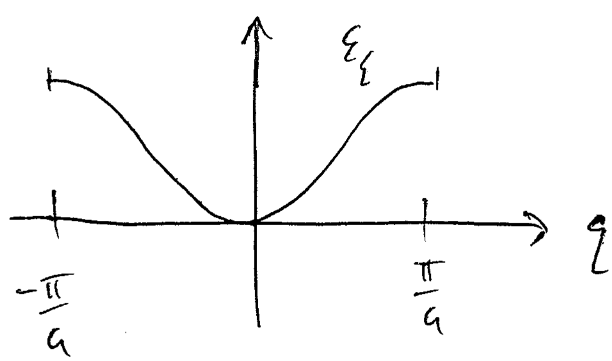
$$\begin{pmatrix} i\varepsilon_q & 2|J|S(1-\cos qa) \\ 2|J|S(\cos qa - 1) & i\varepsilon_q \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0$$

which implies that the matrix must have determinant zero:

$$(i\varepsilon_q)^2 - 2|J|S(1-\cos qa)\,2|J|S(\cos qa - 1) = 0$$

$$\Rightarrow \quad \varepsilon_q^2 = 4|J|^2S^2(1-\cos qa)^2$$

$$\Rightarrow \quad \varepsilon_q = 2|J|S\,|1-\cos qa| \sim 2|J|S\cdot\tfrac{1}{2}(qa)^2 = |J|Sq^2a^2$$



quadratic dispersion near $q=0$

(right- and left-handed polarization $e^{i\varepsilon_q t}$ and $e^{-i\varepsilon_q t}$ associated with the two roots)