Physics 726: Assignment 6

(to be submitted by Thursday, April 14, 2016)

1. The operator $\hat{\boldsymbol{S}} = \frac{1}{2}\sum_{\alpha,\beta} c^\dagger_\alpha \boldsymbol{\sigma} c_\beta$ measures the total intrinsic angular momentum of electrons in an orbital whose occupation is controlled by creation and annihilation operators $c^\dagger$ and c.

   (a) Show explicitly that the four states $|0\rangle = |\text{vac}\rangle$, $|{\uparrow}\rangle = c^\dagger_\uparrow|\text{vac}\rangle$, $|{\downarrow}\rangle = c^\dagger_\downarrow|\text{vac}\rangle$, and $|{\uparrow\downarrow}\rangle = c^\dagger_\uparrow c^\dagger_\downarrow|\text{vac}\rangle$ are eigenstates of $\hat{S}^z$.

   (b) Prove the identity $\hat{S}^2 = \hat{\boldsymbol{S}}\cdot\hat{\boldsymbol{S}} = (3/4)\hat{n}(2-\hat{n})$, where $\hat{n} = \hat{n}_\uparrow + \hat{n}_\downarrow = \sum_\alpha c^\dagger_\alpha c_\alpha$. Use this result to show that the states in part (a) are also eigenstates of $\hat{S}^2$ with eigenvalues 0 or 3/4. *Hint:* first prove that $\hat{n}_\alpha^2 = \hat{n}_\alpha$.

   (c) Argue that $\hat{\boldsymbol{S}}$ behaves as a pure $S = 1/2$ angular momentum when $\sum_\alpha c^\dagger_\alpha c_\alpha = 1$.

2. The rules for the addition of angular momentum can be summarized as

$$S \otimes S' = |S - S'| \oplus \cdots \oplus (S + S' - 1) \oplus (S + S'),$$

   where each spin-S block is $(2S{+}1)$-degenerate. For instance, two spin-half particles combine to give a singlet and a triplet; three combine to give two doublets and a quartet; and so on:

$$\begin{aligned} \tfrac{1}{2}\otimes\tfrac{1}{2} &= 0\oplus 1 \\ \tfrac{1}{2}\otimes\tfrac{1}{2}\otimes\tfrac{1}{2} &= \tfrac{1}{2}\oplus\tfrac{1}{2}\oplus\tfrac{3}{2} \\ \tfrac{1}{2}\otimes\tfrac{1}{2}\otimes\tfrac{1}{2}\otimes\tfrac{1}{2} &= 0\oplus 0\oplus 1\oplus 1\oplus 1\oplus 2 \\ &\;\vdots \end{aligned}$$

   (a) Add the missing entries to the table below.

N	0	1/2	1	3/2	2	5/2	3	total
1		1						2
2	1		1					4
3		2		1				8
4	2		3		1			16
5								
6								
7								
8								
9								

   (b) Consider a ferromagnetic quantum Heisenberg model with long-range interactions:

$$\hat{H} = -2|J|\sum_{i<j}\hat{\boldsymbol{S}}_i\cdot\hat{\boldsymbol{S}}_j.$$

   Suppose that the model is defined on a finite lattice with N sites. Show that the Hamiltonian can be rewritten as

$$\hat{H} = |J|\left[\frac{3N}{4} - \Big(\sum_i \hat{\boldsymbol{S}}_i\Big)^2\right] = |J|\left(\frac{3N}{4} - \hat{S}^2_{\text{tot}}\right).$$

   Determine the ground state energy eigenvalue and ground state degeneracy.

3. The quantum Heisenberg model,

$$\hat{H} = \frac{1}{2}\sum_{j,j'} J_{j,j'}\hat{\boldsymbol{S}}_i \cdot \hat{\boldsymbol{S}}_j = \frac{1}{2}\sum_{j,j'} J_{j,j'} \sum_a \hat{S}^a_j \hat{S}^a_{j'},$$

describes spins on lattice sites j and j' interacting with coupling strength $J_{j,j'}$. (Assume $J_{j,j'} = J_{j',j}$ and $J_{j,j} = 0$.) The sum ranges over $a = x, y, z$. The spin operators obey the product and commutation relations

$$\hat{S}^a_j \hat{S}^b_j = \frac{1}{4}\delta^{a,b}\hat{1} + \frac{i}{2}\sum_c \epsilon^{a,b,c}\hat{S}^c_j,$$

$$[\hat{S}^a_j, \hat{S}^b_{j'}] = \delta_{j,j'}\, i \sum_c \epsilon^{a,b,c}\hat{S}^c_j.$$

(a) Ehrenfest's theorem in this context says that

$$\frac{d}{dt}\langle \hat{\boldsymbol{S}}_j \rangle = i\langle [\hat{H}, \hat{\boldsymbol{S}}_j] \rangle.$$

Evaluate this for the Heisenberg model; you should find that

$$\frac{d}{dt}\langle \hat{\boldsymbol{S}}_j \rangle = \sum_{j' \neq j} J_{j',j} \langle \hat{\boldsymbol{S}}_{j'} \times \hat{\boldsymbol{S}}_j \rangle.$$

Hint: Operator products distribute over the commutator according to $[AB, C] = A[B, C] + [A, C]B$, and $(\boldsymbol{u} \times \boldsymbol{v})^a = \sum_c \epsilon^{a,b,c} u^b v^c$ is the cross product expressed in index notation.

(b) Consider the model on a linear chain, with exchange couplings that connect only nearest-neighbour sites ($J_{j,j+n} = J\delta_{n,1}$). Derive the equation of motion

$$\frac{d}{dt}\langle \hat{\boldsymbol{S}}_j \rangle = J\langle (\hat{\boldsymbol{S}}_{j-1} + \hat{\boldsymbol{S}}_{j+1}) \times \hat{\boldsymbol{S}}_j \rangle.$$

(c) In the large-S limit, $\langle \hat{\boldsymbol{S}}_j \rangle / S \to \boldsymbol{\Omega}$ passes over to a classical unit vector, and the equation of motion obeys

$$\frac{d}{dt}\boldsymbol{\Omega}_j = JS(\boldsymbol{\Omega}_{j-1} + \boldsymbol{\Omega}_{j+1}) \times \boldsymbol{\Omega}_j.$$

Suppose that interactions favour spin alignment ($J = -|J| < 0$) and that the unit vector has the form $\boldsymbol{\Omega}_j = \sqrt{1 - \alpha_j^2 - \beta_j^2}\boldsymbol{e}_z + \alpha_j \boldsymbol{e}_x + \beta_j \boldsymbol{e}_y$, where α_j and β_j represents small local tilts away from the ferromagnetic ground state. Show that the linearlized equations of motion are

$$\frac{d\alpha_j}{dt} = -|J|S(\beta_{j-1} + \beta_{j+1} - 2\beta_j),$$

$$\frac{d\beta_j}{dt} = +|J|S(\alpha_{j-1} + \alpha_{j+1} - 2\alpha_j).$$

(d) Substitute $\alpha_j(t) = \bar{\alpha}\exp[i(qaj - \epsilon_q t)]$ and $\beta_j(t) = \bar{\beta}\exp[i(qaj - \epsilon_q t)]$. Show that the spin wave dispersion relation is $\epsilon_q = 2|J|S(1 - \cos qa)$ and that the excitations have polarization that is either right- or left-handed.

<u>BONUS</u> (e) Work out the antiferromagnetic case ($J > 0$), where $\boldsymbol{\Omega}_j = (-1)^j\sqrt{1 - \alpha_j^2 - \beta_j^2}\boldsymbol{e}_z + \alpha_j \boldsymbol{e}_x + \beta_j \boldsymbol{e}_y$. *Hint:* Don't forget that the antiferromagnetic state breaks translational symmetry.