Physics 726: Assignment 3

(to be submitted by Thursday, February 18, 2016)

1. A system of spinless fermions is described by the Hamiltonian

$$\hat{H} = \sum_{m=1}^{\infty} \varepsilon_m c_m^\dagger c_m,$$

where the energy levels $\varepsilon_1 < \varepsilon_2 < \varepsilon_3 < \cdots$ are strictly ordered. The N-fermion ground state is a *Fermi Sea* with all energy levels filled up to ε_N:

$$|F^{(N)}\rangle = c_N^\dagger \cdots c_2^\dagger c_1^\dagger |\text{vac}\rangle.$$

(a) Show explicitly that $|F^{(N)}\rangle$ is an eigenstate of the total number operator $\hat{N} = \sum_{m=1}^{\infty} c_m^\dagger c_m$.

(b) Confirm that $c_4|F^{(5)}\rangle = -c_5^\dagger c_3^\dagger c_2^\dagger c_1^\dagger|\text{vac}\rangle$, and explain why we should view this as the lowest-lying excited state of the four-fermion system.

(c) Prove that $\langle F^{(N)}|F^{(N')}\rangle = \delta_{N,N'}$, which is to say that the state $|F^{(N)}\rangle$ is (i) properly normalized and (ii) orthogonal to any Fermi Sea with a different number of particles.

(d) Following the logic of question 1(c), demonstrate that

$$\langle F^{(N)}|c_i|F^{(N')}\rangle \sim \delta_{N+1,N'}\,\theta(N \geq 0)\,\theta(N' \geq 1)$$
$$\text{and}\ \ \langle F^{(N)}|c_j^\dagger c_k^\dagger c_l|F^{(N')}\rangle \sim \delta_{N,N'+1}\,\theta(N \geq 2)\,\theta(N' \geq 1).$$

The expressions above rely on a modified Heaviside notation in which $\theta(\text{true}) = 1$ and $\theta(\text{false}) = 0$.

(e) Convince yourself that the first expectation value in question 1(d) vanishes unless $i = N + 1$. List all the possible values of j, k, and l such that the second expectation value is guaranteed to be nonzero.

(f) Show that $\langle F^{(1)}|c_i^\dagger c_j|F^{(1)}\rangle = \delta_{i,1}\delta_{j,1}$ and $\langle F^{(2)}|c_i^\dagger c_j|F^{(2)}\rangle = \delta_{i,1}\delta_{j,1} + \delta_{i,2}\delta_{j,2}$ and that, more generally,

$$\langle F^{(N)}|c_i^\dagger c_j|F^{(N)}\rangle = \sum_{m=1}^{N} \delta_{i,m}\delta_{j,m} = \delta_{i,j}\,\theta(1 \leq i \leq N) = \delta_{i,j}\langle F^{(N)}|\hat{n}_i|F^{(N)}\rangle \equiv \delta_{i,j}\langle \hat{n}_i\rangle.$$

(g) Compute the expectation value of an arbitrary biquadratic operator string. You should find that

$$\langle F^{(N)}|c_i^\dagger c_j^\dagger c_k c_l|F^{(N)}\rangle = (\delta_{i,l}\delta_{j,k} - \delta_{i,k}\delta_{j,l})(1-\delta_{i,j})(1-\delta_{k,l})\langle \hat{n}_i\rangle\langle \hat{n}_j\rangle.$$

(h) The ground state of the three-fermion system is $|F^{(3)}\rangle = c_3^\dagger c_2^\dagger c_1^\dagger|\text{vac}\rangle$. Show that it has energy

$$\sum_m \varepsilon_m \langle F^{(3)}|c_m^\dagger c_m|F^{(3)}\rangle = \varepsilon_1 + \varepsilon_2 + \varepsilon_3.$$

Hint: have a look at what you proved in question 1(f).

(i) Recall that in single-particle quantum mechanics, the wave function $\Psi(x) = \langle x|\Psi\rangle$ is just the position representation of the state $|\Psi\rangle$. With this in mind, compute the real-space, many-body wave function corresponding to the state $|F^{(3)}\rangle$. First, build a state with three fermions in definite positions, $|x_1, x_2, x_3\rangle = \hat{\psi}^\dagger(x_3)\hat{\psi}^\dagger(x_2)\hat{\psi}^\dagger(x_1)|\text{vac}\rangle$. Then take its overlap with the ket of interest:

$$\Psi(x_1, x_2, x_3) = \langle x_1, x_2, x_3|F^{(3)}\rangle = \langle \text{vac}|\hat{\psi}(x_1)\hat{\psi}(x_2)\hat{\psi}(x_3)c_3^\dagger c_2^\dagger c_1^\dagger|\text{vac}\rangle.$$

Hint: use the field operator expansion and remember what you did in question 1(g).

2. Consider a collection of interacting fermions (again, spinless for simplicity). In second quantized form, the Hamiltonian is

$$\hat{H} = \int dx\, \hat{\psi}^\dagger(x) T(x) \hat{\psi}(x) + \frac{1}{2} \int dx \int dy\, \hat{\psi}^\dagger(x) \hat{\psi}^\dagger(y) V(x,y) \hat{\psi}(y) \hat{\psi}(x).$$

Suppose that the one-body term has eigenfunctions $\phi_m(x)$ that satisfy $T\phi_m = \varepsilon_m \phi_m$. Proceed by expressing the field operators as an expansion in these one-body modes:

$$\hat{\psi}(x) = \sum_{m=1}^{\infty} \phi_m(x) c_m \quad \text{and} \quad \hat{\psi}^\dagger(x) = \sum_{m=1}^{\infty} \phi_m^*(x) c_m^\dagger.$$

(a) Show that the Hamiltonian can be written as

$$\hat{H} = \sum_m \varepsilon_m c_m^\dagger c_m + \sum_{j,k,l,m} V_{j,k,l,m} c_j^\dagger c_k^\dagger c_l c_m,$$

in terms of a two-body matrix element

$$V_{j,k,l,m} = \frac{1}{2} \int dx \int dy\, \phi_j^*(x) \phi_k^*(y) V(x,y) \phi_l(y) \phi_m(x).$$

(b) If the interactions are sufficiently weak, we may be able to treat their effect as a perturbation on the noninteracting system, which has a ground state $|F^{(N)}\rangle$. Show that the first-order energy shift is

$$\Delta E = \sum_{i,j,k,l} V_{i,j,k,l} \langle F^{(N)} | c_i^\dagger c_j^\dagger c_k c_l | F^{(N)} \rangle = \sum_{1 \le i < j \le N} 2(V_{i,j,j,i} - V_{i,j,i,j}).$$

(c) Suppose that the fermions interact via a repulsive contact potential $V(x,y) = U a_0 \delta(x-y)$, where U and a_0 are positive constants with units of energy and length. Compute the first-order energy shift for this case. Explain why the answer you get is a direct consequence of the fermions being spinless.