Physics 726: Assignment 1

(to be submitted by Tuesday, February 2, 2016)

1. A system of fermions is described by the Hamiltonian

$$\hat{H} = \sum_{\alpha=1}^{3} \varepsilon_\alpha c_\alpha^\dagger c_\alpha,$$

where the energy levels obey the ordering $\varepsilon_1 < \varepsilon_2 < \varepsilon_3$. The creation and annihilation operators, c and $c^\dagger$ obey the usual anticommutation relations,

$$\{c_\alpha, c_\beta^\dagger\} = c_\alpha c_\beta^\dagger + c_\beta^\dagger c_\alpha = \delta_{\alpha,\beta}.$$

   (a) The zero-particle ground state is $|\psi_0^{(0)}\rangle = |\text{vac}\rangle$, and the one-particle ground state is $|\psi_0^{(1)}\rangle = c_1^\dagger|\text{vac}\rangle$. Write down the ground state $|\psi_0^{(2)}\rangle$, first excited state $|\psi_1^{(2)}\rangle$, and second excited $|\psi_2^{(2)}\rangle$ state of the two-particle system.

   (b) Argue that $c_\alpha|\text{vac}\rangle$ and $c_\alpha^\dagger|\psi_0^{(3)}\rangle$ both vanish.

   (c) Ensure that $|\psi_0^{(2)}\rangle$ is properly normalized.

   (d) Show explicitly that the energy eigenequation

$$\hat{H}|\psi_0^{(2)}\rangle = (\varepsilon_1 + \varepsilon_2)|\psi_0^{(2)}\rangle$$

   is satisfied. *Hint:* use the anticommutator and apply the property that $c_\alpha|\text{vac}\rangle = 0$.

2. A system of bosons is described by the Hamiltonian

$$\hat{H} = \sum_{\alpha=1}^{3} \varepsilon_\alpha a_\alpha^\dagger a_\alpha.$$

As before, $\varepsilon_1 < \varepsilon_2 < \varepsilon_3$. The creation and annihilation operators, a and $a^\dagger$ obey the usual commutation relations,

$$[a_\alpha, a_\beta^\dagger] = a_\alpha a_\beta^\dagger - a_\beta^\dagger a_\alpha = \delta_{\alpha,\beta}.$$

   (a) The zero-particle ground state is $|\psi_0^{(0)}\rangle = |\text{vac}\rangle$, and the one-particle ground state is $|\psi_0^{(1)}\rangle = a_1^\dagger|\text{vac}\rangle$. Write down the ground states $|\psi_0^{(2)}\rangle$ and $|\psi_0^{(3)}\rangle$ of the two- and three-particle system.

   (b) Normalize $|\psi_0^{(1)}\rangle$, $|\psi_0^{(2)}\rangle$, and $|\psi_0^{(3)}\rangle$. Solve by recursion—or just guess—the normalization factor for $|\psi_0^{(n)}\rangle$.

   (c) Show explicitly that

$$\hat{H}|\psi_0^{(3)}\rangle = 3\varepsilon_1|\psi_0^{(3)}\rangle.$$