Physics 7330: Assignment 3

(to be submitted by Wednesday, September 30, 2026)

I invite you to attempt Assignment 3 and to submit all your work for Questions 1 and 2. Please turn in a paper copy in class. You're welcome to use Mathematica to help you with any part of the assignment. If you do, please email to `kbeach@olemiss.edu` a single Wolfram Notebook, sent as an attachment, that contains any relevant Mathematica code. Use the naming convention `Phys7330-A3-webid.nb`, and include `Phys7330-Fall2026-webid Assignment 3 Submission` on the subject line. (Be sure to replace `webid` with your own personal UM WebID; mine, for instance, is `kbeach`.) If you end up doing the entire assignment by hand, just turn in a paper copy and ignore the email attachment instructions.

Formalism refresher

You are probably most familiar with three-dimensional, Euclidean space. Consider a vector $\boldsymbol{v} \in \mathbb{R}^3$. With respect to a basis set $\{\boldsymbol{e}_i : i = 1, 2, 3 \text{ or } i = x, y, z\}$ that spans $\mathbb{R}^3$ and is othonormal ($\boldsymbol{e}_i \cdot \boldsymbol{e}_j = \delta_{i,j}$), the vector has coordinates $\{v_i = \boldsymbol{e}_i \cdot \boldsymbol{v}\}$. We sometimes casually write "$\boldsymbol{v} = (v_1, v_2, v_3)$" or "$\boldsymbol{v} = (v_x, v_y, v_z)$" to mean $\boldsymbol{v} = \sum_i v_i \boldsymbol{e}_i = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3 = v_x\boldsymbol{e}_x + v_y\boldsymbol{e}_y + v_z\boldsymbol{e}_z$. Of course, given a different orthonormal basis set $\{\boldsymbol{e}'_i\}$, the vector would have a different triplet of coordinates $\{v'_i = \boldsymbol{v} \cdot \boldsymbol{e}'_i\}$. The sets $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}'_i = \sum_j R_{i,j}(\boldsymbol{n}, \theta)\boldsymbol{e}'_j\}$ can transform one into the other via rotation.

Elements of the same linear vector space in Dirac notation are written as $|v\rangle$, which is just a label adorned by a vertical pipe on the left and an angled bracket on the right. This object (which in LaTeX is typeset as `\vert v \rangle`) is called a *ket*. The space is spanned by the basis $\{|1\rangle, |2\rangle, |3\rangle\}$ (which we assume to be orthonormal: $\langle i|j\rangle = \delta_{i,j}$), and hence the coordinates of v are the inner products $v_1 = \langle 1|v\rangle$, $v_2 = \langle 2|v\rangle$, and $v_3 = \langle 3|v\rangle$. Another perspective is that there exists a resolution of unity, $\hat{1} = \sum_i |i\rangle\langle i|$, such that

$$|v\rangle = \hat{1}|v\rangle = \left(\sum_i |i\rangle\langle i|\right)|v\rangle = \sum_i |i\rangle\langle i|v\rangle = \sum_i v_i|i\rangle = v_1|1\rangle + v_2|2\rangle + v_3|3\rangle.$$

As a consistency check, notice that the second component is

$$\langle 2|v\rangle = v_1 \cancelto{0}{\langle 2|1\rangle} + v_2 \cancelto{1}{\langle 2|2\rangle} + v_3 \cancelto{0}{\langle 2|3\rangle}.$$

In a different basis, $\{|i'\rangle\}$, the vector is $|v\rangle = \hat{1}'|v\rangle = \sum_i |i'\rangle\langle i'|v\rangle = v'_1|1'\rangle + v'_2|2'\rangle + v'_3|3'\rangle$. The basis transformation (unprimed to primed) is the operator

$$\hat{1}'\hat{1} = \left(\sum_i |i'\rangle\langle i'|\right)\left(\sum_j |j\rangle\langle j|\right) = \sum_{i,j} |i'\rangle\langle i'|j\rangle\langle j| = \sum_{i,j} |i'\rangle U_{i,j}\langle j|,$$

whose kernel is the matrix $U_{i,j} = \langle i'|j\rangle$.

In the context of quantum mechanics, we generalize to a complex vector space of arbitrary dimension (the Hilbert space $\mathcal{H}$). Accordingly, $\alpha^*\langle i|$ is a vector dual to $\alpha|i\rangle$; the *inner* product of $\alpha|i\rangle$ and $\beta|j\rangle$ is $\alpha^*\beta\langle i|j\rangle$, a complex number; and the *outer* product of $\alpha|i\rangle$ and $\beta|j\rangle$ is $\alpha\beta^*|i\rangle\langle j|$, an operator that maps kets into kets. In our notation, $|i\rangle, |j\rangle \in \mathcal{H}$ are kets (state vectors) in the Hilbert space, $\langle i|, \langle j| \in \mathcal{H}^\star$ are the corresponding bras in the space dual to $\mathcal{H}$, and $\alpha, \beta \in \mathbb{C}$. Self-consistency requires that inner products obey the following complex conjugation rule:

$$\Big(\big(\alpha^*\langle i|\big)\big(\beta|j\rangle\big)\Big)^* = \big(\alpha^*\beta\langle i|j\rangle\big)^* = \alpha\beta^*\langle j|i\rangle = \big(\beta^*\langle j|\big)\big(\alpha|i\rangle\big).$$

Outer products acted on by the adjoint or hermitian conjugate ($\dagger$) transform as follows:

$$\Big(\big(\alpha|i\rangle\big)\big(\beta^*\langle j|\big)\Big)^\dagger = \big(\alpha\beta^*|i\rangle\langle j|\big)^\dagger = \alpha^*\beta|j\rangle\langle i| = \big(\beta|j\rangle\big)\big(\alpha^*\langle i|\big).$$

An abstract operator $\hat{O}$ can be put it into a particular representation by sandwiching the operator between (otherwise inert) representations of unity:

$$\hat{O} = \hat{1}\hat{O}\hat{1} = \sum_{i,j} |i\rangle\langle i|\hat{O}|j\rangle\langle j| = \sum_{i,j} |i\rangle O_{i,j}\langle j|.$$

Its hermitian conjugate is therefore

$$\hat{O}^\dagger = \sum_{i,j} |j\rangle O^*_{i,j}\langle i| = \sum_{i,j} |i\rangle O^*_{j,i}\langle j|.$$

We understand the matrix elements in this representation to be $O_{i,j} = \langle i|\hat{O}|j\rangle$ and $O^*_{j,i} = \langle i|\hat{O}^\dagger|j\rangle$, the complex-conjugate transpose of $O_{i,j}$.

Assignment questions

1. A linear vector space is spanned by three orthonomal vectors, denoted by the kets $|a\rangle$, $|b\rangle$, and $|c\rangle$. A state $|\psi\rangle$ is constructed as the linear combination $|\psi\rangle = 2\alpha|a\rangle - i\beta|b\rangle + \gamma|c\rangle$, where $\alpha, \beta, \gamma \in \mathbb{C}$.

   (a) Show that the inner product (or "overlap") of the ψ state with itself is

   $$\langle\psi|\psi\rangle = 4|\alpha|^2 + |\beta|^2 + |\gamma|^2$$

   and that

   $$|\tilde{\psi}\rangle = \frac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}} = \frac{2\alpha|a\rangle - i\beta|b\rangle + \gamma|c\rangle}{\sqrt{4|\alpha|^2 + |\beta|^2 + |\gamma|^2}}.$$

   is a properly normalized state.

   (b) Suppose that the system is prepared in the state $|\psi\rangle$ and we perform an experiment to determine if the system is found in microstate a, b, or c (mutually exclusive after a measurement). What is the classical probability of finding the system in microstate b? *Hint:* You can understand this as the expectation value of the projection operator $\hat{P}_b = |b\rangle\langle b|$ with respect to $|\psi\rangle$.

   (c) Prove that $\hat{1} = |a\rangle\langle a| + |b\rangle\langle b| + |c\rangle\langle c|$ is the identity operator for the vector space.

   (d) For the cyclic permutation operator $\hat{T} = |b\rangle\langle a| + |c\rangle\langle b| + |a\rangle\langle c|$, compute the expectation value $\langle\tilde{\psi}|\hat{T}|\tilde{\psi}\rangle = \langle\psi|\hat{T}|\psi\rangle/\langle\psi|\psi\rangle$. You should be able to show that

   $$\langle\tilde{\psi}|\hat{T}|\tilde{\psi}\rangle = \frac{2\alpha^*\gamma + 2i\beta^*\alpha - i\gamma^*\beta}{4|\alpha|^2 + |\beta|^2 + |\gamma|^2}.$$

   (e) Find the eigenstates and corresponding eigenvalues of $\hat{T}$. Proceed by computing the matrix elements in the *abc* basis:

   $$T = \begin{pmatrix} \langle a|\hat{T}|a\rangle & \langle a|\hat{T}|b\rangle & \langle a|\hat{T}|c\rangle \\ \langle b|\hat{T}|a\rangle & \langle b|\hat{T}|b\rangle & \langle b|\hat{T}|c\rangle \\ \langle c|\hat{T}|a\rangle & \langle c|\hat{T}|b\rangle & \langle c|\hat{T}|c\rangle \end{pmatrix}$$

```
T = {{0, 0, 1}, {1, 0, 0}, {0, 1, 0}}
{Evals, Evecs} = Eigensystem[T]
Evecs = Normalize /@ Evecs
```

(f) Find the unitary transformation matrix V such that

$$V^\dagger T V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 + i\sqrt{3}/2 & 0 \\ 0 & 0 & -1/2 - i\sqrt{3}/2 \end{pmatrix}$$

```
V = Transpose[Evecs]
Expand[Transpose[Conjugate[V]].T.V]
```

(g) Prove that $\hat{T}^{-1} = \hat{T}^\dagger$ is unitary and that $\hat{T}^3 = \hat{1}$.

```
Inverse[T] == Transpose[Conjugate[T]]
T.T.T == IdentityMatrix[3]
```

2. Consider the 4×4 matrix

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Suppose that a quantum system has the Hamiltonian matrix $H = h + h^\dagger$.

```
h = ArrayReshape[Range[16], {4, 4}]
H = h + Transpose[h]
H // MatrixForm
```

(a) Express the matrix elements $h_{i,j}$ as a function of the rows $i = 1, 2, 3, 4$ and columns $j = 1, 2, 3, 4$. Find an analogous index-notation expression for $H_{i,j}$.

(b) Consider the operator

$$\hat{H} = \sum_{i=1}^{4} \sum_{j=1}^{4} |i\rangle H_{i,j} \langle j|$$

that acts on vectors in the linear vector space of kets. First verify that $H_{i,j} = \langle i|\hat{H}|j\rangle$. Then prove that two applications of the operator are equivalent to

$$\hat{H}^2 = \sum_{i=1}^{4} \sum_{j=1}^{4} |i\rangle \left(\sum_{k=1}^{4} H_{i,k} H_{k,j} \right) \langle j|.$$

```
H.H == Table[Sum[H[[i,k]]*H[[k,j]],{k,1,4}],{i,1,4},{j,1,4}]
```

(c) Solve for the energy eigenstates and corresponding energy eigenvalues.

```
{Evals, Evecs} = Eigensystem[H]
```

Argue that $|\phi_0\rangle = 2|1\rangle - 3|2\rangle + |4\rangle$ and $|\phi_0'\rangle = |1\rangle - 2|2\rangle + |3\rangle$ are states of zero energy. Show explicitly that $\hat{H}$ annihilates both kets.

(d) For the operator $\hat{N}$ obeying $\hat{N}|n\rangle = n|n\rangle$, compute the expectation values of $\hat{N}$ with respect to $|\phi_0\rangle$ and $|\phi_0'\rangle$. As a check, note that these have to take on a value between 1 and 4.