Physics 7330: Assignment 2

(to be submitted by Wednesday, September 16, 2026)

I invite you to attempt Assignment 2 and to submit all your work for Questions 1 and 2. Please turn in a paper copy in class. You're welcome to use Mathematica to help you with any part of the assignment. If you do, please email to `kbeach@olemiss.edu` a single Wolfram Notebook, sent as an attachment, that contains any relevant Mathematica code. Use the naming convention `Phys7330-A2-webid.nb`, and include `Phys7330-Fall2026-webid Assignment 2 Submission` on the subject line. (Be sure to replace `webid` with your own personal UM WebID; mine, for instance, is `kbeach`.) If you end up doing the entire assignment by hand, just turn in a paper copy and ignore the email attachment instructions.

1. The Baker–Campbell–Hausdorff formula is an infinite expansion for the product of the exponentiation of two noncommuting operators A and B. (Note that I am electing not to use any special notation to decorate the operators; just infer from the context, and use your judgement.) The first few terms are shown:

$$e^A e^B = \exp\Big(A + B + \frac{1}{2}[A,B] + \frac{1}{12}[A,[A,B]] + \frac{1}{12}[B,[B,A]] + \cdots\Big).$$

In quantum mechanics applications, this is often expressed as

$$e^{\lambda(A+B)} = e^{\lambda A} e^{\lambda B} \exp\Big(-\frac{\lambda^2}{2}[A,B] + O(\lambda^3)\Big).$$

   (a) Evaluate the commutator $[p^2, V(x)]$. The potential $V(x)$ is a smooth function of position x, and the momentum $p = (\hbar/i)(d/dx)$ is a differential operator.

   (b) Consider the one-dimensional Hamiltonian $H = p^2/2m + V(x)$, describing a quantum particle of mass m moving in the potential $V(x)$. Argue that the evolution operator $U(t) = e^{-iHt/\hbar}$ satisfies $i\hbar(\partial/\partial t)U(t) = HU(t)$ and that it is *unitary*—which here means $U(t)^\dagger = U(t)^{-1} = U(-t)$.

   (c) Given $H = p^2/2m + V(x)$ and your result in Question 1(a), derive an expression for the evolution operator over short times, up to $O(t^3)$.

   (d) Suppose that we want to approximate the quantum evolution over an arbitrary interval $[0,t]$. We can always view the evolution as occurring in N consecutive steps,

$$U(t) = \underbrace{U(\delta t)U(\delta t)\cdots U(\delta t)}_{N \text{ factors}},$$

   where $\delta t = t/N$ is very brief (provided that N is chosen suitably large). Argue convincingly that exact evolution can be recovered in the limit

$$U(t) = \lim_{N\to\infty} \big(e^{-ip^2\delta t/2m\hbar} e^{-iV(x)\delta t/\hbar}\big)^N.$$

   Explain why this is not the same as $e^{-ip^2t/2m\hbar}e^{-iV(x)t/\hbar}$.

2. In class we considered a planar quantum rotor model with a symmetry-breaking term that favours angular orientation near $\phi = 0$:

$$H = \frac{L^2}{2I} - I\Omega^2 \cos\phi.$$

Here, I is the moment of inertia, and Ω is the natural frequency of the corresponding classical problem in the small-amplitude-oscillation limit. The operators ϕ and L obey the canonical commutation relationship $[\phi, L] = i\hbar$. We made the decision to work in the ϕ-representation, so that the operators take the form ϕ and $L = (\hbar/i)\partial/\partial\phi$ and act on a wave function $\psi(\phi)$.

   (a) Show that states $\chi_m(\phi) \sim \exp(im\phi)$ are eigenstates of the angular momentum operator with eigenvalue $\hbar m$. Determine the proper normalization of the states. Here's a Mathematica version of the solution:

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\[Chi][\[Phi]_] = Exp[I m \[Phi]]/Sqrt[2 \[Pi]]
Assuming[m \[Element] Integers, Integrate[Conjugate[\[Chi][\[Phi]]] \[Chi][\[Phi]],
    {\[Phi], 0, 2 \[Pi]}]]
L = (\[HBar]/I) D[#, \[Phi]] &
L[\[Chi][\[Phi]]]/\[Chi][\[Phi]]
```

   (b) Argue that the parity operation (reflection across the preferred axis, $\phi \to -\phi$) is a symmetry of the Hamiltonian. Construct a basis of states of definite even ($P = +1$) and odd ($P = -1$) parity from linear combinations of the angular momentum states $\chi_m(\phi)$. Explain how this basis can be used to block diagonalize the Hamiltonian.

   (c) Consider a truncated basis that contains only the *two lowest-lying states* in each of the even- and odd-parity sectors. Write the time-independent Schrödinger equation as two 2×2 matrix eigenvector problems. You should be able to show that the relevant matrix blocks are

$$H^{(+)} = \begin{pmatrix} 0 & \frac{I\Omega^2}{\sqrt{2}} \\ \frac{I\Omega^2}{\sqrt{2}} & \frac{\hbar^2}{2I} \end{pmatrix} = \epsilon_0 \begin{pmatrix} 0 & -\sqrt{2}g \\ -\sqrt{2}g & 1 \end{pmatrix},$$

$$H^{(-)} = \begin{pmatrix} \frac{\hbar^2}{2I} & \frac{I\Omega^2}{2} \\ \frac{I\Omega^2}{2} & \frac{2\hbar^2}{I} \end{pmatrix} = \epsilon_0 \begin{pmatrix} 1 & -g \\ -g & 4 \end{pmatrix},$$

   where $\epsilon_0 = \hbar^2/2I$ is an energy scale.

   (d) Solve the even- and odd-parity 2×2 eigenproblems. Report the energy eigenvalues and corresponding eigenvectors. (You are welcome to make use of Mathematica's `Eigensystem` or its `Eigenvectors` and `Eigenvalues` commands.) You should find that an even parity state with energy

$$E_0^{(+)} = \frac{1}{2}\left(1 - \sqrt{1 + 8g^2}\right)$$

   is the state of lowest eenergy state across all values of the dimensionless coupling $g = (I\Omega/\hbar)^2$. For that lowest energy state—viz., the *ground state*—plot the probability density $|\psi_0^{(+)}(\phi)|^2$ of finding the rotor in the vicinity of angle ϕ. Do this for small, intermediate, and large values of g.

   (e) Compute the expectation values of H with respect to the next-lowest-lying states in each sector. Based on energy comparisons, comment on the appropriateness of the basis truncation.