Physics 7330: Assignment 5

(to be submitted by Wednesday, November 11, 2026)

I invite you to attempt Assignment 5 and to submit all your work for Questions 1–5. Please turn in a paper copy in class. You're welcome to use Mathematica to help you with any part of the assignment. If you do, please email to `kbeach@olemiss.edu` a single Wolfram Notebook, sent as an attachment, that contains any relevant Mathematica code. Use the naming convention `Phys7330-A5-webid.nb`, and include `Phys7330-Fall2026-webid Assignment 5 Submission` on the subject line. (Be sure to replace `webid` with your own personal UM WebID; mine, for instance, is `kbeach`.) If you end up doing the entire assignment by hand, just turn in a paper copy and ignore the email attachment instructions.

The Hamiltonian of a one-dimensional quantum harmonic oscillator is given by the differential operator

$$H_0 = -\frac{\hbar^2}{2m}\frac{d}{dx^2} + \frac{1}{2}m\omega_0^2 x^2.$$

The wave functions

$$\psi_n^{(0)}(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m\omega x^2}{2\hbar}\right)\mathsf{H}_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right), \text{ with } n = 0, 1, 2, \ldots,$$

are its energy eigenfunctions, satisfying $H_0\psi_n^{(0)}(x) = E_n^{(0)}\psi_n^{(0)}(x) = \hbar\omega_0(n + 1/2)\psi_n^{(0)}(x)$. The first four Hermite polynomials (in the physicists' convention) are:

$$\mathsf{H}_0(s) = 1, \quad \mathsf{H}_1(s) = 2s, \quad \mathsf{H}_2(s) = 4s^2 - 2, \quad \mathsf{H}_3(s) = 8s^3 - 12s.$$

Note the even/oddness of $\psi_n^{(0)}(-x) = (-1)^n\psi_n^{(0)}(x)$, which is inherited from $\mathsf{H}_n(-s) = (-1)^n\mathsf{H}_n(s)$.

Now consider what happens if we add an attractive delta-function spike at the origin:

$$H = H_0 + V = -\frac{\hbar^2}{2m}\frac{d}{dx^2} + \frac{1}{2}m\omega_0^2 x^2 - V_0 a\delta(x).$$

For convenience, let's work in units where $m = \hbar = \omega_0 = 1$ and $V_0 a = \upsilon$.

1. Evaluating the Hermite polynomials at the origin gives $\mathsf{H}_{2k+1} = 0$ and $\mathsf{H}_{2k} = (-1)^k(2k)!/k!$, for non-negative integer k. Find an explicit expression for $|\psi_n^{(0)}(0)|^2$, distinguishing even n and odd n.

2. Argue that, for odd n, the energy is $E_n = n + 1/2$ (exactly, to all orders) and that for even n,

$$E_n = n + \frac{1}{2} - \upsilon|\psi_n(0)|^2 + \upsilon^2|\psi_n(0)|^2\sum_{m\neq n}\frac{|\psi_m(0)|^2}{n-m} - O(\upsilon^3)$$

3. Show that the ground state energy goes as

$$E_0 = \frac{1}{2} - \frac{\upsilon}{\sqrt{\pi}} - \frac{\upsilon^2\ln 2}{\pi} - O(\upsilon^3).$$

   You can make use of the fact that $\sum_{k=1}^{\infty} 4^{-k}(k!)^{-2}(2k-1)! = \ln 2$.

4. Explain why $E_0 < E_0^{(0)}$ and why E_0 is a monotonically decreasing function of υ.

5. Show that the ground state wave function (unnormalized) is

$$\psi_0(x) = \psi_n^{(0)} + \frac{\upsilon}{\sqrt{\pi}}\sum_{k=1}^{\infty}\frac{(-1)^k\sqrt{(2k)!}}{2^k\,(2k)\,k!}\psi_{2k}^{(0)}(x) - O(\upsilon^2)$$

   Without performing any calculation (just using your intuition and good quantum sense), sketch the ground state wave function in the presence of the attractive delta-function spike.