Physics 711: Assignment 3

(to be submitted by Wednesday, October 1, 2025)

I invite you to attempt Assignment 3 and to submit all your work for Questions 1(a–e) and 2(a–e). Please turn in a paper copy in class. Also email to kbeach@olemiss.edu a single Wolfram Notebook, sent as an attachment, that contains any relevant Mathematica code. Use the naming convention `Phys711-A3-webid.nb`, and be sure to include `Phys711-Fall2025-webid Assignment 3 Submission` on the subject line. (Be sure to replace `webid` with your own personal UM WebID; mine, for instance, is `kbeach`.)

1. The commutator of operators $\hat{A}$ and $\hat{B}$ is defined as $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. (Note that version 14.3 of Mathematica introduces the commands `Commutator` and `NonCommutativeExpand`.)

   (a) Prove that $[\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}]$ and $[\hat{A}, \hat{A}] = 0$.

   (b) For operators $\hat{A}$, $\hat{B}$, and $\hat{C}$ and complex numbers α and β, show that

   $$[\alpha\hat{A} + \beta\hat{B}, \hat{C}] = \alpha[\hat{A}, \hat{C}] + \beta[\hat{B}, \hat{C}].$$

   (c) Prove that the commutator distributes over an operator product as follows:

   $$[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}].$$

   (d) Prove the famous Jacobi identity:

   $$[\hat{A}, [\hat{B}, \hat{C}]] + [\hat{B}, [\hat{C}, \hat{A}]] + [\hat{C}, [\hat{A}, \hat{B}]] = 0.$$

   (e) Consider operators $\hat{x}$ and $\hat{p}$ satisfying $[\hat{x}, \hat{p}] = i\hbar$. Argue that $[\hat{x}, \hat{p}^2] = 2i\hbar\hat{p}$, $[\hat{x}, \hat{p}^3] = 3i\hbar\hat{p}^2$, and more generally, $[\hat{x}, \hat{p}^n] = ni\hbar\hat{p}^{n-1}$. Use this result to show that

   $$[\hat{x}, f(\hat{p})] = i\hbar f'(\hat{p}),$$

   where f is any smooth function.

2. In class we considered a planar quantum rotor model with a symmetry-breaking term that favours angular orientation near $\phi = 0$:

   $$\hat{H} = \frac{\hat{L}^2}{2I} - I\Omega^2 \cos\hat{\phi}.$$

   Here, I is the moment of inertia, and Ω is the natural frequency of the corresponding classical problem in the small-amplitude-oscillation limit. The operators $\hat{\phi}$ and $\hat{L}$ obey the canonical commutation relationship $[\hat{\phi}, \hat{L}] = i\hbar$. We made the decision to work in the ϕ-representation, so that the operators take the form ϕ and $L = (\hbar/i)\partial/\partial\phi$ and act on a wave function $\psi(\phi)$.

   (a) Show that states $\chi_m(\phi) \sim \exp(im\phi)$ are eigenstates of the angular momentum operator with eigenvalue $\hbar m$. Determine the proper normalization of the states. Here's a Mathematica version of the solution:

   ```
   \[Chi][\[Phi]_] = Exp[I m \[Phi]]/Sqrt[2 \[Pi]]
   Assuming[m \[Element] Integers, Integrate[Conjugate[\[Chi][\[Phi]]] \[Chi][\[Phi]],
       {\[Phi], 0, 2 \[Pi]}]]
   L = (\[HBar]/I) D[#, \[Phi]] &
   L[\[Chi][\[Phi]]]/\[Chi][\[Phi]]
   ```

   (b) Argue that the parity operation (reflection across the preferred axis, $\phi \to -\phi$) is a symmetry of the Hamiltonian. Construct a basis of states of definite even ($P = +1$) and odd ($P = -1$) parity from linear combinations of the angular momentum states $\chi_m(\phi)$. Explain how this basis can be used to block diagonalize the Hamiltonian.

(c) Consider a truncated basis that contains only the two lowest-lying states in each of the even- and odd-parity sectors. Write the time-independent Schrödinger equation as two 2×2 matrix eigenvector problems.

(d) Solve the even-parity 2×2 eigenproblem. (You are welcome to make use of Mathematica's `Eigensystem` or its `Eigenvectors` and `Eigenvalues` commands.) For the ground state, plot the probability density $|\psi(\phi)|^2$ of finding the rotor in the vicinity of angle ϕ. Do this for small, intermediate, and large values of Ω.

(e) Compute the expectation values of $\hat{H}$ with respect to the next-lowest-lying states in each sector. Based on energy comparisons, comment on the appropriateness of the basis truncation.