# Physics 711: Assignment 2

(to be submitted by Wednesday, September 17, 2025)

I invite you to attempt Assignment 2 and to submit all your work for Questions 1(a–e), 2, and 3. Please turn in a paper copy in class. Also email to `kbeach@olemiss.edu` a single Wolfram Notebook, sent as an attachment, that contains the relevant Mathematica code. Use the naming convention `Phys711-A2-webid.nb`, and be sure to include `Phys711-Fall2025-webid Assignment 2 Submission` on the subject line. (Be sure to replace `webid` with your own personal UM WebID; mine, for instance, is `kbeach`.)

1. Three states have energy $\epsilon$, $2\epsilon$, and $3\epsilon$. Small hopping amplitudes of strength $-t$ connect states 1 and 2 and states 2 and 3.

$$H = \begin{pmatrix} \epsilon & -t & 0 \\ -t & 2\epsilon & -t \\ 0 & -t & 3\epsilon \end{pmatrix}$$

   (a) Show that the energy eigenvalues are $E_1 = 2\epsilon$, $E_2 = 2\epsilon - \sqrt{2t^2+\epsilon^2}$, and $E_3 = 2\epsilon + \sqrt{2t^2+\epsilon^2}$. Try doing this by hand and then verifying with the `Eigenvalues` command.

   (b) Find the matrix $V$ with normalized columns such that

$$HV = V\begin{pmatrix} 2\epsilon & 0 & 0 \\ 0 & 2\epsilon - \sqrt{2t^2+\epsilon^2} & 0 \\ 0 & 0 & 2\epsilon + \sqrt{2t^2+\epsilon^2} \end{pmatrix}.$$

   Here, $V = (v_1|v_2|v_3)$ is the matrix formed by stitching together as columns the three eigenvectors that satisfy $Hv_n = E_n v_n$ (for $n = 1, 2, 3$). That is to say, $\sum_j H_{i,j}(v_n)_j = E_n (v_n)_i$ or

$$\sum_j H_{i,j} V_{j,n} = (v_n)_i E_n = \sum_j V_{i,j} E_n \delta_{j,n},$$

   where $V_{j,n} = (v_n)_j$ is the $j$th component of the $n$th eigenvector. Don't attempt this by hand. Make use of the `Eigenvectors` or `Eigensystem` commands.

   (c) Prove that $\operatorname{tr} H = 6\epsilon$ and $\operatorname{tr} H^2 = 4t^2 + 14\epsilon^2$.

   (d) Find a closed-form expression for $\operatorname{tr} H^k$, where $k = 1, 2, 3, \ldots$ is a positive integer. You should be able to apply the binomial theorem to show that

$$\operatorname{tr} H^k = \epsilon^k \Big[1 + \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{2(k!)}{4^j (2j)!(k-2j)!}\Big(1 + \frac{2t^2}{\epsilon^2}\Big)^j\Big].$$

   (e) Prove these two closely related identities:

$$\operatorname{tr} e^{iHt/\hbar} = e^{i2\epsilon t/\hbar}\Big(1 + 2\cos\big[(2t^2+\epsilon^2)^{1/2} t/\hbar\big]\Big),$$
$$\operatorname{tr} e^{-\beta H} = e^{-2\beta\epsilon}\Big(1 + 2\cosh\big[\beta(2t^2+\epsilon^2)^{1/2}\big]\Big).$$

2. The operators

$$\hat{P} = \sum_{i,j=1,2} |i\rangle P_{i,j} \langle j| = \big(|1\rangle\ |2\rangle\big)\, P \begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix}$$

$$\text{and}\ \ \hat{Q} = \sum_{i,j=1,2} |i\rangle Q_{i,j} \langle j| = \big(|1\rangle\ |2\rangle\big)\, Q \begin{pmatrix} \langle 1| \\ \langle 2| \end{pmatrix}$$

are defined by the matrix kernels $P = \vec{p} \cdot \vec{\sigma}$ and $Q = \vec{q} \cdot \vec{\sigma}$, which have elements $P_{i,j} = p^a \sigma^a_{i,j}$ and $Q_{i,j} = q^a \sigma^a_{i,j}$. (Because of the repeated index, there is an implied summation over $a = x, y, z$.)

Show that

$$[P, Q] = 2i(\vec{p} \times \vec{q}) \cdot \vec{\sigma} \ \ \text{and} \ \ [\hat{P}, \hat{Q}] = 2i\epsilon^{abc} p^a q^q \sigma^c_{i,j} |i\rangle\langle j|.$$

Furthermore, argue that $\hat{P}$ and $\hat{Q}$ commute iff $\vec{p}$ and $\vec{q}$ are colinear.

3. Let's prove the the operator $e^{i\hat{p}\xi/\hbar}$ is a *generator of translations* in real space. In particular show that

$$\langle x| e^{i\hat{p}\xi/\hbar} |\psi\rangle = \psi(x + \xi),$$

where $\langle x|\psi\rangle = \psi(x)$ is the real-space representation (i.e., the wave function) of the state $|\psi\rangle$. You may find it helpful to expand the exponentiated operator as a powerseries in $\xi$ and then to re-sum the series. You can make use of the result $\langle x|\hat{p}|\psi\rangle = -i\hbar(d/dx)\langle x|\psi\rangle = -i\hbar\psi'(x)$.