Physics 711: Assignment 1

(to be submitted by Friday, September 5, 2025)

I invite you to attempt Assignment 1 and to submit all your work for Questions 1(a–g) and 2(a–d). Please turn in a paper copy in class. Also email to kbeach@olemiss.edu a single Wolfram Notebook, sent as an attachment, that contains the relevant Mathematica code. Use the naming convention `Phys711-A1-webid.nb`, and be sure to include `Phys711-Fall2025-webid Assignment 1 Submission` on the subject line. (Be sure to replace `webid` with your own personal UM WebID; mine, for instance, is `kbeach`.)

1. A linear vector space is spanned by three orthonomal vectors, denoted by the kets $|a\rangle$, $|b\rangle$, and $|c\rangle$. A state $|\psi\rangle$ is constructed as the linear combination $|\psi\rangle = \alpha|a\rangle + \beta|b\rangle + \gamma|c\rangle$, where $\alpha, \beta, \gamma \in \mathbb{C}$. The corresponding state in the dual space is $\langle\psi| = \alpha^*\langle a| + \beta^*\langle b| + \gamma^*\langle c|$.

   (a) Show that the inner product (or "overlap") is
   $$\langle\psi|\psi\rangle = |\alpha|^2 + |\beta|^2 + |\gamma|^2$$
   and that
   $$|\tilde{\psi}\rangle = \frac{|\psi\rangle}{\sqrt{\langle\psi|\psi\rangle}} = \frac{\alpha|a\rangle + \beta|b\rangle + \gamma|c\rangle}{\sqrt{|\alpha|^2 + |\beta|^2 + |\gamma|^2}}.$$
   is a properly normalized state.

   (b) Suppose that the system is prepared in the state $|\psi\rangle$ and we perform an experiment to determine if the system is found in microstate a, b, or c (mutually exclusive after a measurement). What is the classical probability of finding the system in microstate b? *Hint:* You can understand this as the expectation value of the filtering operator $\hat{P}_b = |b\rangle\langle b|$ with respect to $|\psi\rangle$.

   (c) Prove that $\hat{\mathbb{1}} = |a\rangle\langle a| + |b\rangle\langle b| + |c\rangle\langle c|$ is the identity operator for the vector space.

   (d) For the cyclic permutation operator $\hat{T} = |b\rangle\langle a| + |c\rangle\langle b| + |a\rangle\langle c|$, compute the expectation value $\langle\tilde{\psi}|\hat{T}|\tilde{\psi}\rangle = \langle\psi|\hat{T}|\psi\rangle/\langle\psi|\psi\rangle$. You should be able to show that
   $$\langle\tilde{\psi}|\hat{T}|\tilde{\psi}\rangle = \frac{\alpha^*\gamma + \beta^*\alpha + \gamma^*\beta}{|\alpha|^2 + |\beta|^2 + |\gamma|^2}.$$

   (e) Find the eigenstates and corresponding eigenvalues of $\hat{T}$. Proceed by computing the matrix elements in the *abc* basis:
   $$T = \begin{pmatrix} \langle a|\hat{T}|a\rangle & \langle a|\hat{T}|b\rangle & \langle a|\hat{T}|c\rangle \\ \langle b|\hat{T}|a\rangle & \langle b|\hat{T}|b\rangle & \langle b|\hat{T}|c\rangle \\ \langle c|\hat{T}|a\rangle & \langle c|\hat{T}|b\rangle & \langle c|\hat{T}|c\rangle \end{pmatrix}$$

   ```
   T = {{0, 1, 0}, {0, 0, 1}, {1, 0, 0}}
   {Evals, Evecs} = Eigensystem[T]
   Evecs = Normalize /@ Evecs
   ```

   (f) Find the unitary transformation matrix V such that
   $$V^\dagger T V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 + i\sqrt{3}/2 & 0 \\ 0 & 0 & -1/2 - i\sqrt{3}/2 \end{pmatrix}$$

   ```
   V = Transpose[Evecs]
   Expand[Transpose[Conjugate[V]].T.V]
   ```

(g) Prove that $\hat{T}^{-1} = \hat{T}^\dagger$ is unitary and that $\hat{T}^3 = \hat{\mathbb{1}}$.

```
Inverse[T] == Transpose[Conjugate[T]]
T.T.T == IdentityMatrix[3]
```

2. Consider the 4×4 matrix

$$h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{pmatrix}.$$

Suppose that a quantum system has the Hamiltonian matrix $H = h + h^\dagger$.

```
h = ArrayReshape[Range[16], {4, 4}]
H = h + Transpose[h]
H // MatrixForm
```

(a) Express the matrix elements $h_{i,j}$ as a function of the rows $i = 1, 2, 3, 4$ and columns $j = 1, 2, 3, 4$. Find an analogous index-notation expression for $H_{i,j}$.

(b) Consider the operator

$$\hat{H} = \sum_{i=1}^{4} \sum_{j=1}^{4} |i\rangle H_{i,j} \langle j|$$

that acts on vectors in the linear vector space of kets. First verify that $H_{i,j} = \langle i|\hat{H}|j\rangle$. Then prove that two applications of the operator are equivalent to

$$\hat{H}^2 = \sum_{i=1}^{4} \sum_{j=1}^{4} |i\rangle \left(\sum_{k=1}^{4} H_{i,k} H_{k,j} \right) \langle j|.$$

```
H.H == Table[Sum[H[[i,k]]*H[[k,j]],{k,1,4}],{i,1,4},{j,1,4}]
```

(c) Solve for the energy eigenstates and corresponding energy eigenvalues.

```
{Evals, Evecs} = Eigensystem[H]
```

Argue that $|\phi_0\rangle = 2|1\rangle - 3|2\rangle + |4\rangle$ and $|\phi_0'\rangle = |1\rangle - 2|2\rangle + |3\rangle$ are states of zero energy. Show explicitly that $\hat{H}$ annihilates both kets.

(d) For the operator $\hat{N}$ obeying $\hat{N}|n\rangle = n|n\rangle$, compute the expectation values of $\hat{N}$ with respect to $|\phi_0\rangle$ and $|\phi_0'\rangle$. As a check, note that these have to take on a value between 1 and 4.