Physics 651: Exercise 4

(not for submission)

To begin, we summarize the key results of vector calculus that we discussed in class. In our notation, f is a scalar field and $\mathbf{F}$ is a vector field; the nabla symbol denotes the gradient (viz., $\nabla = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$ in rectangular coordinates, $\nabla = \hat{\rho}\partial_\rho + (1/\rho)\hat{\phi}\partial_\phi + \hat{z}\partial_z$ in cylindrical polar); and ∂R represents the boundary of some region R.

- The *fundamental theorem for line integrals* states that

  $$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_2) - f(\mathbf{r}_1),$$

  where C is a directed contour from $\mathbf{r}_1$ to $\mathbf{r}_2$. The result depends only on the starting and end points and not on the particular path taken by C. (It explains why forces derived from a potential must obey an energy conservation law.)

- The *divergence theorem* (also known as Gauss's theorem or Ostrogradsky's theorem) states that

  $$\int_V \nabla \cdot \mathbf{F}\, dV = \int_{\partial V} \mathbf{F} \cdot d\mathbf{S},$$

  where dV is a volume element, and $d\mathbf{S} = \hat{n}dS$ is the directed surface element pointing to the exterior of V. This result connects the charges contained in V to the flux through its boundary surface.

- *Stoke's theorem* states that

  $$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

  Here, S is an open surface, and ∂S is a contour along the boundary of S directed in a right-hand sense with respect to the orientation of $d\mathbf{S}$. This result connects the circulation of a field on the surface to the field's net contribution around the surface's edge.

1. Consider the vector field

   $$\mathbf{F} = \mathbf{F}(\rho, \phi, z) = \frac{\rho\hat{\rho}\sin^2\phi + \rho(\cos\phi\sin\phi)\hat{\phi} + z\hat{z}}{(z^2 + \rho^2\sin^2\phi)^{3/2}},$$

   expressed in cylindrical polar coordinates.

   (a) Explain why the position $\mathbf{r} = \rho\hat{\rho} + z\hat{z}$ has a differential $d\mathbf{r} = (d\rho)\hat{\rho} + \rho(d\phi)\hat{\phi} + (dz)\hat{z}$.

   (b) Consider the line integral along a contour C that can be parameterized by $\rho(t) = \ell t$, $\phi(t) = \pi t/2$, $z(t) = \ell\cos\pi t$ with t ranging from 0 to 1. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 dt\, \cdots$ explicitly to obtain $(2 - \sqrt{2})/2\ell$.

   (c) Find a scalar field $V(\mathbf{r})$ such that $\mathbf{F} = -\nabla V$.

   (d) Now use the fundamental theorem for line integrals to confirm that

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{2 - \sqrt{2}}{2\ell}.$$

2. Verify Stokes' theorem for the vector field $\mathbf{F}(x, y, z) = -xy\hat{x} + x\hat{y} + xz\hat{z}$ and surface S, where S is a 2×2 square patch centred on the origin with corners at $(-1, -1, 0)$ and $(1, 1, 0)$.

We follow the Fourier transform convention used in the *Physical Mathematics* textbook:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \tilde{f}(k) e^{ikx} = \mathcal{F}^{-1}[\tilde{f}(k)](x).$$
$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} = \mathcal{F}[f(x)](k).$$

The default convention in *Mathematica* is almost the same, but the signs in the arguments of the exponentials are swapped ($e^{ikx} \leftrightarrow e^{-ikx}$). We can use the `FourierParameters` option to modify the convention to match that of the textbook. For convenience, let's define shortcut functions:

```
FT[f_, x_, k_] := FourierTransform[f, x, k, FourierParameters -> {0, -1}]
IFT[ff_, k_, x_] := InverseFourierTransform[ff, k, x, FourierParameters -> {0, -1}]
```

Complete the following questions using any combination of analytical and computer-assisted (Mathematica) solution methods.

3. Explore the following Fourier transform pairs.

```
FT[1, x, k]
IFT[DiracDelta[x], x, k]

g[x_] = Sqrt[a] Exp[-x^2/2 a^2];
Assuming[a > 0, FT[g[x], x, k]]
Assuming[a > 0, Simplify[g[x] /. {a -> 1/a, x -> k}]]

FT[Sqrt[2 Pi] Sinc[x], x, k]
FT[HeavisideTheta[x + 1] - HeavisideTheta[x - 1], x, k]

FT[Sqrt[2 Pi] HeavisideTheta[x] Exp[-a x], x, k]
Assuming[a > 0, IFT[1/(a + I k), k, x]]
```

4. Here we verify the convolution theorem, $\widetilde{f * g} = \tilde{f}\tilde{g}$, with the definition

$$(f * g)(x) = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} f(x-y) g(y),$$

in the case where $f(x) = g(x) = \theta(x + 1/2) - \theta(x - 1/2)$ are both square bump functions of unit area. Note that Mathematica's `Convolve` does not include the $1/\sqrt{2\pi}$ factor, which has to be put in by hand. Verify the theorem for other choices of $f(x) \neq g(x)$.

```
Convolve[UnitBox[y], UnitBox[y], y, x]/Sqrt[2 Pi]
FT[%, x, k]
FT[UnitBox[x], x, k]^2
```