Physics 651: Exercise 3

(not for submission)

1. The determinant of an $n \times n$ matrix A is given by

$$\det A = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_n=1}^{n} \epsilon_{j_1,j_2,\ldots,j_n} A_{1,j_1} A_{2,j_2} \cdots A_{n,j_n}$$

or

$$(\det A)\, \epsilon_{i_1,i_2,\ldots,i_n} = \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_n=1}^{n} \epsilon_{j_1,j_2,\ldots,j_n} A_{i_1,j_1} A_{i_2,j_2} \cdots A_{i_n,j_n} = \sum_{\{j_1,\ldots,j_n\}} \epsilon_{j_1,j_2,\ldots,j_n} \prod_{k=1}^{n} A_{i_k,j_k},$$

where ϵ is the Levi-Civita symbol. For the $n = 4$ case, only one of the following terms appears in the sum. Which one?

   (a) $+A_{1,1}A_{2,2}A_{3,4}A_{4,3}$

   (b) $-A_{1,3}A_{2,1}A_{3,4}A_{4,2}$

   (c) $+A_{1,1}A_{2,2}A_{3,1}A_{4,2}$

   (d) $-A_{1,3}A_{2,3}A_{3,3}A_{4,3}$

   If you would like to see all $5! = 120$ terms written out, execute these *Mathematica* commands:

```
detA = Expand[Det[Table[Subscript[A, i, j], {i, 1, 5}, {j, 1, 5}]]]
Length[detA] == 5!
anotherDetA = Total[Map[Signature[#] Product[Subscript[A, k, #[[k]]], {k, 1, 5}] &,
    Permutations[Range[5]]]]
detA == anotherDetA
```

2. The set of vectors $\{\mathbf{v}_1 = \hat{x} + 2\hat{y}, \mathbf{v}_2 = \hat{x} + \hat{y} + \hat{z}, \mathbf{v}_3 = -\hat{y} + 3\hat{z}\}$ spans $\mathbb{R}^3$ but does not constitute an orthonormal set. Normalize the vectors and apply the Gram-Schmidt procedure. Once you've generated the set

$$\left\{\mathbf{u}_1 = \frac{1}{\sqrt{5}}(\hat{x} + 2\hat{y}), \mathbf{u}_2 = \cdots, \mathbf{u}_3 = \cdots\right\},$$

verify explicitly that $\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{i,j}$ for all $i \leq j$. Try all of this by hand, but then check your answers against the output of this *Mathematica* code:

```
u = Orthogonalize[{{1, 2, 0}, {1, 1, 1}, {0, -1, 3}}]
Table[u[[i]].u[[j]], {i, 1, 3}, {j, 1, 3}] // MatrixForm
```

3. Consider an arbitrary 2×2 matrix, A.

   (a) Express its characteristic polynomial, $\mathcal{P}_A(\lambda) = \det(\lambda I - A)$, in terms of $\operatorname{tr} A$ and $\det A$.

   (b) Show that A has two eigenvalues,

$$\lambda_\pm = \frac{1}{2}\operatorname{tr} A \pm \sqrt{\frac{1}{4}(\operatorname{tr} A)^2 - \det A}.$$

Here is an example of how to automate the calculations in 3(a) and 3(b):

```
A = {{a, b}, {c, d}}
thisTrA = Tr[A]
thisDetA = Det[A]
PP[x_] = Collect[CharacteristicPolynomial[A, x], -x]
P[x_] = PP[x] /. thisDetA -> DetA /. thisTrA -> TrA
Roots[P[x] == 0, x]
soln = Solve[P[x] == 0, x]
{SubPlus[\[Lambda]], SubMinus[\[Lambda]]} = x /. soln
```

4. The linear operator $\hat{A}$ is expressed as $\sum_{i,j} |i\rangle A_{i,j} \langle j|$ in some generic basis and as $\sum_k |\phi_k\rangle \lambda_k \langle \phi_k|$ in the basis of eigenstates (which satisfy $\hat{A}|\phi_k\rangle = \lambda_k |\phi_k\rangle$). We assume that A is hermitian and that both $\{|i\rangle\}$ and $\{|\phi_k\rangle\}$ are orthonormal sets. Which of the following statements is incorrect?
   (a) $U_{j,k} = \langle j|\phi_k\rangle$ describes the unitary matrix that transforms between the two basis sets.
   (b) The eigenstate basis diagonalizes A with the eigenvalues $\{\lambda_k\}$ all taking real values.
   (c) The kth eigenvalue can be extracted from $\lambda_k = (U^\dagger A U)_{k,k}$
   (d) The ith eigenvalue can be extracted from $\lambda_i = \langle i|\hat{A}|i\rangle$

5. The column vectors
$$v^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } v^{(2)} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$
are eigenvectors of the matrix
$$A = \begin{pmatrix} 1 & 1 \\ 3 & -1 \end{pmatrix}.$$
   (a) Find the corresponding eigenvalues, λ_1 and λ_2.
   (b) Prove that $v^{(1)}$ and $v^{(2)}$ are linearly independent.
   (c) Compute the matrix elements $S_{a,b} = v^{(a)T} v^{(b)} = \sum_k v_k^{(a)} v_k^{(b)}$ and explain why $S_{a,b} \neq \delta_{a,b}$.
   (d) Find the transformation matrix V such that
$$A = V \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V^{-1}$$
   (e) Compute $\exp A$

You can also try this:

```
A = {{1, 1}, {3, -1}}
v1 = {1, 1}
v2 = {-1, 3}
evals = First[Solve[{A . v1 == \[Lambda]1 v1, A . v2 == \[Lambda]2 v2}, {\[Lambda]1,
    \[Lambda]2}]]
Sort[Eigenvalues[A]] == Sort[{\[Lambda]1, \[Lambda]2} /. evals]
V = Transpose[{v1, v2}]
V // MatrixForm
Inverse[V] . A . V
A == V . diag . Inverse[V]
MatrixExp[A]
V . ExpDiag . Inverse[V] == MatrixExp[A]
```

6. The column vectors

$$v^{(1)} = \begin{pmatrix} 1+\sqrt{2} \\ -1 \end{pmatrix} \quad \text{and} \quad v^{(2)} = \begin{pmatrix} 1-\sqrt{2} \\ -1 \end{pmatrix}$$

are eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}.$$

(a) Find the corresponding eigenvalues, λ_1 and λ_2.

(b) Prove that $v^{(1)}$ and $v^{(2)}$ are linearly independent.

(c) Compute the matrix elements $S_{a,b} = v^{(a)T} v^{(b)} = \sum_k v_k^{(a)} v_k^{(b)}$ and explain why $S_{a,b} \neq \delta_{a,b}$. What can you do to $v^{(1)}$ and $v^{(2)}$ to ensure that their overlap matrix elements give exactly the Kronecker delta?

(d) Find the transformation matrix V such that

$$A = V \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} V^{\dagger}$$

(e) Show that

$$\sin(\theta A) = \begin{pmatrix} \sin 2\theta & -\sin\theta \\ -\sin\theta & 0 \end{pmatrix}.$$

Questions 6(a)–(e) can be automated as follows:

```
A = {{2, -1}, {-1, 0}}
{{e1, e2}, {v1, v2}} = Eigensystem[A]
Det[{v1, v2}]
S = Simplify[{{v1.v1, v1.v2}, {v2.v1, v2.v2}}]
v1 = v1/Sqrt[v1.v1]
v2 = Normalize[v2]
S = Simplify[{{v1 . v1, v1 . v2}, {v2 . v1, v2 . v2}}]
V = Transpose[{v1, v2}]
V // MatrixForm
Simplify[Inverse[V]] // MatrixForm
Simplify[V.{{e1, 0}, {0, e2}}.Simplify[Inverse[V]]]
Sin[\[Theta] A]
```

7. We want to extremize the function $f(x, y) = xy^3$ simultaneously on the two curves $3x^4 + y^4 = 1$ and $x^4 + 3y^4 = 1$. Which of the following is a correct statement?

(a) The minimum value is $-1/4$ and the maximum value is $+1/4$.

(b) The minimum value is 0 and the maximum value is $+1/2$.

(c) The minimum value is $-1/2$ and the maximum value is unbounded.

(d) The minimum value is unbounded and the maximum value is $1/2$.

8. Let's work with the vector field

$$\mathbf{F} = \mathbf{F}(\mathbf{r}) = \mathbf{F}(x, y, z) = \frac{4x^3\hat{x} - 4y^3\hat{y} + 2z\hat{z}}{(x^4 - y^4 + z^2)^3},$$

expressed in rectangular coordinates.

(a) Consider the line integral along a contour C that lies in the $z = 2$ plane and that is the bounding box to the square formed by the intersection of the lines $|x| = 1$ and $|y| = 1$. Substitute this specific contour parameterization into

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \frac{4x^3\,dx - 4y^3\,dy + 2z\,dz}{(x^4 - y^4 + z^2)^3}.$$

Show explicitly that the integral evaluates to zero.

(b) Find a scalar field $V(\mathbf{r})$ such that $\mathbf{F} = -\nabla V$.

(c) The *fundamental theorem for line integrals* states that

$$\int_{\mathbf{r}_1}^{\mathbf{r}_2} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_2) - f(\mathbf{r}_1),$$

regardless of the path taken from $\mathbf{r}_1$ to $\mathbf{r}_2$. Use the fundamental theorem for line integrals to argue that this integral from part (a) must vanish:

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0.$$