Physics 651: Exercise 2

(not for submission)

1. The kets $|u\rangle$, $|v\rangle$, and $|w\rangle$ belong to a vector space that is spanned by the orthonormal basis $\{|b_i\rangle\}$. Let $\hat{P} = \sum_{i,j} |b_i\rangle P_{i,j} \langle b_j|$ and $\hat{Q} = \sum_{i,j} |b_i\rangle Q_{i,j} \langle b_j|$ be linear operators acting on that space. Which of the following expressions is incorrect?
   (a) $\langle u|\hat{P}|v\rangle^* = \langle v|\hat{P}^\dagger|u\rangle$
   (b) $\big(|u\rangle\langle v|\big)|w\rangle = \langle u|v\rangle|w\rangle$
   (c) $\langle b_i|\hat{P}\hat{Q}|u\rangle = \sum_{j,k} P_{i,j} Q_{j,k} u_k$
   (d) $\big(\hat{P}\hat{Q}|u\rangle\big)^\dagger = \langle u|\hat{Q}^\dagger\hat{P}^\dagger$

2. Let α, β, and γ be complex numbers and $|u\rangle$, $|v\rangle$, and $|w\rangle$ be elements of a complex vector space. Which of the following expressions is correct?
   (a) $\big(|u\rangle\langle v|\big)|w\rangle = \langle v|w\rangle|u\rangle$
   (b) $\langle u|\big(|v\rangle\langle w|\big) = \langle u|v\rangle^*|w\rangle$
   (c) $\big(\alpha|u\rangle \otimes |v\rangle \otimes |w\rangle\big)^\dagger = \alpha^*\langle w| \otimes \langle v| \otimes \langle u|$
   (d) $\big(\alpha|u\rangle + \beta|v\rangle + \gamma|w\rangle\big)^\dagger = \alpha\langle u| + \beta\langle v| + \gamma\langle w|$

3. Associated with a quantum system in its ground state $|\psi\rangle$ is a density operator $\hat{\rho} = |\psi\rangle\langle\psi|$. When expressed in terms of a particular basis $\{|n\rangle\}$, the ground state has component amplitudes $\psi_n = \langle n|\psi\rangle$. For an observable $\hat{O}$, having matrix elements $\langle m|\hat{O}|n\rangle = O_{m,n}$, the ground state expectation value is

$$\langle \hat{O} \rangle = \frac{\operatorname{tr} \hat{\rho}\hat{O}}{\operatorname{tr} \hat{\rho}}.$$

   Show that this is equivalent to

$$\frac{\sum_{m,n} \psi_m^* O_{m,n} \psi_n}{\sum_k |\psi_k|^2}.$$

4. The determinant of a 2×2 matrix A is given by

$$\det A = \sum_{i=1}^{2} \sum_{j=1}^{2} \epsilon_{i,j} A_{1,i} A_{2,j}.$$

   What is the correct definition of the alternating symbol?
   (a) $\epsilon_{1,1} = \epsilon_{2,2} = 0$ and $\epsilon_{1,2} = \epsilon_{2,1} = 1$
   (b) $\epsilon_{1,1} = \epsilon_{2,2} = 0$ and $\epsilon_{1,2} = -\epsilon_{2,1} = 1$
   (c) $\epsilon_{1,1} = \epsilon_{2,2} = 1$ and $\epsilon_{1,2} = \epsilon_{2,1} = -1$
   (d) $\epsilon_{1,1} = -\epsilon_{2,2} = 1$ and $\epsilon_{1,2} = \epsilon_{2,1} = 0$

5. The determinant of a 4×4 matrix A is given by

$$\det A = \sum_{i=1}^{4} \sum_{j=1}^{4} \sum_{k=1}^{4} \sum_{l=1}^{4} \epsilon_{i,j,k,l} A_{1,i} A_{2,j} A_{3,k} A_{4,l},$$

where $\epsilon_{i,j,k,l}$ is the 4-index Levi-Civita symbol. Which one of the following terms appears in the sum.

   (a) $+A_{1,1}A_{2,2}A_{3,4}A_{4,3}$
   (b) $-A_{1,3}A_{2,1}A_{3,4}A_{4,2}$
   (c) $+A_{1,1}A_{2,2}A_{3,1}A_{4,2}$
   (d) $-A_{1,3}A_{2,3}A_{3,3}A_{4,3}$

6. Here, $|u\rangle$ and $|v\rangle$ are elements of a vector space; $\hat{A}$, $\hat{B}$, and $\hat{C}$ are linear operators acting on the space; and $\{|i\rangle\}$ constitutes an orthonormal basis for the space. Use the technique of inserting representations of unity, $\hat{1} = \sum_i |i\rangle\langle i|$, to prove that

$$\langle u|\hat{A}\hat{B}\hat{C}|v\rangle^* = \langle v|\hat{C}^\dagger \hat{B}^\dagger \hat{A}^\dagger |u\rangle.$$

7. Rotation about the x, y, and z axes (in the right-hand sense about the directions $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$) is implemented by matrices

$$R_1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \ R_2(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \ R_3(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

   (a) Show that $\det R_i(\theta) = 1$ for each of $i = 1, 2, 3$ and for all values of the angle θ.
   (b) Prove that $R_i(-\theta) = R_i(\theta)^T = R_i(\theta)^{-1}$.
   (c) Evaluate these three composite rotations:

$$A = R_1(-\pi/2)R_2(\pi/2)R_1(\pi/2),$$
$$B = R_3(\pi/2)R_2(\pi/4)R_1(\pi/2),$$
$$C = R_1(-\pi/4)R_3(\pi/2)R_1(\theta)R_3(-\pi/2)R_1(\pi/4).$$

   In other words, evaluate each of the matrix products to determine the resulting 3×3 matrix.

   (d) Prove that A corresponds to a rotation about $\mathbf{e}_3$; B to a rotation about $\mathbf{e}_1 + (1 + \sqrt{2})\mathbf{e}_2 + \mathbf{e}_3$; and C to a rotation about $-\mathbf{e}_2 + \mathbf{e}_3$. *Hint:* One way to think about this is that if matrix R describes a rotation about an axis parallel to the column vector e, then R must leave e invariant; i.e., $Re = e$, which says that e is an eigenvector of R with eigenvalue 1. If this holds for $R = R(\theta)$, then the same must be true for $R(-\theta) = R^{-1} = R^T$. Hence, to determine the axes of rotation, we can solve the eigenproblems for $(A + A^T)/2$, $(B + B^T)/2$, and $(C + C^T)/2$, and pick out the eigenvectors with eigenvalue 1. Alternatively, we can look for the vectors that are annihilated by $A - A^T$, $B - B^T$, and $C - C^T$.

(e) Note that $1 + 2\cos\theta = \operatorname{tr} R_3(\theta) = \operatorname{tr} \tilde{R}^{-1}\tilde{R}R_3(\theta) = \operatorname{tr} \tilde{R}R_3(\theta)\tilde{R}^{-1}$, for an arbitrary rotation $\tilde{R}$. This tells us that the trace always reveals the rotation angle. Use $\operatorname{tr} A = 1 + 2\cos\theta_A = 1$ and $\operatorname{tr} B = 1 + 2\cos\theta_B = 1/\sqrt{2}$ to determine the angles of rotation. Evaluate $\operatorname{tr} C = 1 + 2\cos\theta$ to confirm that the parameter θ does in fact represent the rotation angle.

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R1[\[Theta]_] := { { 1, 0, 0}, {0, Cos[\[Theta]], -Sin[\[Theta]]}, {0, Sin[\[Theta]],
    Cos[\[Theta]]}}
R2[\[Theta]_] := { { Cos[\[Theta]], 0, Sin[\[Theta]]}, {0, 1, 0}, {-Sin[\[Theta]], 0,
    Cos[\[Theta]]}}
R3[\[Theta]_] := { { Cos[\[Theta]], -Sin[\[Theta]], 0}, {Sin[\[Theta]], Cos[\[Theta]],
    0}, {0, 0, 1}}
A = R1[-\[Pi]/2] . R2[\[Pi]/2] . R1[\[Pi]/2]
B = R3[\[Pi]/2] . R2[\[Pi]/4] . R1[\[Pi]/2]
CC = R1[-\[Pi]/4] . R3[\[Pi]/2] . R1[\[Theta]] . R3[-\[Pi]/2] . R1[\[Pi]/4]
Solve[Tr[A] == 1 + 2 Cos[\[Theta]], \[Theta]]
Solve[Tr[A] == 1 + 2 Cos[\[Theta]], \[Theta]]
Solve[Tr[A] == 1 + 2 Cos[\[Theta]], \[Theta]]
EVA = Simplify[Eigensystem[(A + Transpose[A])/2]]
EVB = Simplify[Eigensystem[(B + Transpose[B])/2]]
EVC = Simplify[Eigensystem[(CC + Transpose[CC])/2]]
MemberQ[EVA[[1]], 1]
For[i = 1, i <= 3, ++i, If[EVA[[1]][[i]] == 1, Print[EVA[[2]][[i]]]]]
MemberQ[EVB[[1]], 1]
For[i = 1, i <= 3, ++i, If[EVB[[1]][[i]] == 1, Print[EVB[[2]][[i]]]]]
MemberQ[EVC[[1]], 1]
For[i = 1, i <= 3, ++i, If[EVC[[1]][[i]] == 1, Print[EVC[[2]][[i]]]]]
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(f) Reflection across the $x = 0$ plane is represented by the matrix

$$M = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

This maps every column vector

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{to a reflected vector} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ y \\ z \end{pmatrix}.$$

Use a similarity transformation to determine the matrix $M' = U^{-1}MU$ that reflects across the plane defined by $y = -x$.