Physics 651: Assignment 5

(to be submitted by Tuesday, October 29, 2024)

I invite you to attempt Assignment 5 and to turn in your solutions to Questions 1–2 and 5–8. Any handwritten derivations should be submitted to me in hard copy (or as a scan in `pdf` format). Any computational results should be collected in a single Wolfram Notebook and sent as an attachment to `kbeach@olemiss.edu`. Please follow the naming convention `Phys651-A5-webid.nb`, and be sure to include the subject line `Phys651-Fall2024-webid Assignment 5 Submission`.

1. Prove the following:

   (a) For 3-vectors **a**, **b**, **c**,

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{i,j,k}a_i b_j c_k = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b};$$

   (b) $\nabla\cdot(\nabla\times\mathbf{a}) = 0$;

   (c) $\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$.

   Hint: Some of these equalities are more easily proved in tensor notation by applying properties of the Levi-Civita symbol.

2. The function $\pi^2 - x^2$ peaks at $x = 0$ and falls off to zero at $x = \pm\pi$. Its 2π-periodic extension can be constructed from a cosine Fourier series (with an infinite number of terms). The following code demonstrates the series cut off after just five terms.

```
bump[x_] = \[Pi]^2 - x^2
Plot[bump[x], {x, -\[Pi], \[Pi]}]
FourierCosCoefficient[bump[x], x, n]
bump5[x_] = FourierCosSeries[bump[x], x, 5]
Plot[{bump[x], bump[x - 2 \[Pi]], bump[x + 2 \[Pi]], bump5[x]}, {x, -2.1 \[Pi], 2.1 \[Pi]},
    Frame -> True, PlotRange -> {{-7, 7}, {-2, 11}}]
```

   (a) Use *Mathematica* to create a log-linear plot of the discrepancy (in absolute value) between $\pi^2 - x^2$ and its 50-term cosine series approximation. You should find that the errors are smaller than 10^{-4} near $x = 0$. Justify the shape of resulting curve; in particular, explain why the error is so much larger near $x = \pm\pi$?

   (b) Now consider $(\pi^2 - x^2)(1 + \cos^2 10x)$ on the interval $[-\pi, \pi]$. Show that its the 20-term Fourier expansion is radically more faithful than the 19-term expansion. Then explain why.

3. We can use the ceiling operation to create a sawtooth wave $\lceil x\rceil - x$ of height 1 and period 1. The stretched function of period 2π is $\lceil x/2\pi\rceil - x/2\pi$. We can observe the convergence of its Fourier series as follows:

```
sawtooth[x_] = Ceiling[x/(2 \[Pi])] - x/(2 \[Pi])
saw2th[x_] = Sum[f[n] Exp[I n x], {n, -\[Infinity], \[Infinity]}]
Plot[{sawtooth[x], saw2th[x]}, {x, -15, 15}]
f[n_] = FourierCoefficient[sawtooth[x], x, n]
Plot[{Sum[f[n] Exp[I n x], {n, -8, 8}], sawtooth[x]}, {x, -15, 15}]
Plot[{Sum[f[n] Exp[I n x], {n, -16, 16}], sawtooth[x]}, {x, -15, 15}]
Plot[{Sum[f[n] Exp[I n x], {n, -32, 32}], sawtooth[x]}, {x, -15, 15}]
```

   Now consider the function $\lceil x/\pi\rceil - x/\pi$. This new function actually has period π, but if we treat it as having period 2π and perform the Fourier analysis as above, what changes? Comment on the properties of the sawtooth wave's Fourier series coefficient, as represented by `f[n]`. Note its parity, and its behaviour at $n = 0$ and for even and odd n.

4. Suppose that $A(\xi)$, $B(\xi)$, and $C(\xi)$ are functions of ξ that obey $AB = C$. Suppose further that A and C are known and that we need to solve for B. Prove that the most general solution is

$$B = \frac{C}{A} + h\,\delta(A),$$

where $h(\xi)$ is an arbitrary function.

5. Consider a scalar field $\phi(x, t)$, with one space and one time argument, governed by the partial differential equation

$$\partial_t^2\phi + 2\gamma\partial_t\phi - c^2\partial_x^2\phi = f.$$

The additional symbols denote the forcing function $f(x, t)$, damping coefficient γ, and speed c. (This is just a twist on the wave equation example we did in class.) Substitute the Fourier representation

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, \tilde{\phi}(k,\omega),\, e^{i(kx-\omega t)}$$

and then show that $\tilde{\phi}$ must satisfy

$$-(\omega^2 + 2i\gamma\omega - c^2k^2)\tilde{\phi}(k,\omega) = \tilde{f}(k,\omega).$$

6. Argue that the most general solution is

$$\tilde{\phi}(k,\omega) = \tilde{h}(k,\omega)\delta(P(\omega)) - \frac{\tilde{f}(k,\omega)}{P(\omega)},$$

where $P(\omega) = \omega^2 + 2i\gamma\omega - c^2k^2$ is a quadratic polynomial in the frequency variable. (*Hint:* Think back to what you proved in question 4.)

7. Solve for the roots of $P(\omega)$:

$$\omega_\pm = -i\gamma \pm \sqrt{c^2k^2 - \gamma^2} \equiv -i\gamma \pm \omega_k.$$

Prove that the most general solution in space and time (having inverse transformed $k, \omega \to x, t$) is of the form

$$\phi(x,t) = e^{-\gamma t}\int_{-\infty}^{\infty} dk\, e^{ikx}\big(Ae^{-i\omega_k t} + Be^{i\omega_k t}\big) - \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega\, \frac{\tilde{f}(k,\omega)e^{i(kx-\omega t)}}{\omega^2 + 2i\gamma\omega - c^2k^2}.$$

8. Impose the forcing function $f(x,t) = \exp(-x^2/2\xi^2)\cos\Omega t$ and take the long time limit $t \gg \gamma$. (We have introduced two constants: ξ is a length, and Ω is a frequency.) Show that the asymptotic behaviour of the field is given by

$$\phi(x,t) = -\frac{\xi}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dk \frac{(\Omega^2 - c^2k^2)\cos\Omega t - 2\gamma\Omega\sin\Omega t}{(\Omega^2 - c^2k^2)^2 + 4\gamma^2\Omega^2}(\cos kx)e^{-\frac{1}{2}k^2\xi^2}.$$

9. Suppose that $\phi(x,t)|_{t=0} = p(x)$ and $\partial_t\phi(x,t)|_{t=0} = q(x)$ are known initial conditions and that the forcing function is also known and nonzero. Roughly sketch out how you would determine $\phi(x,t)$ for all future $t > 0$. (If you're feeling adventurous, you might want to select a specific form for each of $p(x)$ and $q(x)$ and plot up time snapshots of the field's time evolution in Mathematica. One straightforward choice is a plucked string: $p(x)$ gaussian and $q(x) = 0$.)