Physics 651: Exercise 4

(not for submission)

We follow the Fourier transform convention used in the *Physical Mathematics* textbook:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk\, \tilde{f}(k) e^{ikx} = \mathcal{F}^{-1}[\tilde{f}(k)](x).$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx\, f(x) e^{-ikx} = \mathcal{F}[f(x)](k).$$

The default convention in *Mathematica* is almost the same, but the signs in the arguments of the exponentials are swapped ($e^{ikx} \leftrightarrow e^{-ikx}$). We can use the `FourierParameters` option to modify the convention to match that of the textbook. For convenience, let's define shortcut functions:

```
FT[f_, x_, k_] := FourierTransform[f, x, k, FourierParameters -> {0, -1}]
IFT[ff_, k_, x_] := InverseFourierTransform[ff, k, x, FourierParameters -> {0, -1}]
```

Complete the following questions using any combination of analytical and computer-assisted (Mathematica) solution methods.

1. Suppose that $A(\xi)$, $B(\xi)$, and $C(\xi)$ are functions of ξ that obey $AB = C$. Suppose further that A and C are known and that we need to solve for B. Prove that the most general solution is

$$B = \frac{C}{A} + h\,\delta(A),$$

where $h(\xi)$ is an arbitrary function.

2. Consider a scalar field $\phi(x,t)$, with one space and one time argument, governed by the partial differential equation

$$\partial_t^2 \phi + 2\gamma \partial_t \phi - c^2 \partial_x^2 \phi = f.$$

The additional symbols denote the forcing function $f(x,t)$, damping coefficient γ, and speed c. (This is just a twist on the wave equation example we did in class.) Substitute the Fourier representation

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}}\, \tilde{\phi}(k,\omega), e^{i(kx-\omega t)}$$

and then show that $\tilde{\phi}$ must satisfy

$$-(\omega^2 + 2i\gamma\omega - c^2k^2)\tilde{\phi}(k,\omega) = \tilde{f}(k,\omega).$$

3. Argue that the most general solution is

$$\tilde{\phi}(k,\omega) = \tilde{h}(k,\omega)\delta(P(\omega)) - \frac{\tilde{f}(k,\omega)}{P(\omega)},$$

where $P(\omega) = \omega^2 + 2i\gamma\omega - c^2k^2$ is a quadratic polynomial in the frequency variable. (*Hint:* Think back to what you proved in question 1.)

4. Solve for the roots of $P(\omega)$:

$$\omega_{\pm} = -i\gamma \pm \sqrt{c^2k^2 - \gamma^2} \equiv -i\gamma \pm \omega_k.$$

Prove that the most general solution in space and time (having inverse transformed $k,\omega \to x,t$) is of the form

$$\phi(x,t) = e^{-\gamma t} \int_{-\infty}^{\infty} dk\, e^{ikx}\Big(Ae^{-i\omega_k t} + Be^{i\omega_k t}\Big) - \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega\, \frac{\tilde{f}(k,\omega)e^{i(kx-\omega t)}}{\omega^2 + 2i\gamma\omega - c^2k^2}.$$

5. Impose the forcing function $f(x,t) = \exp(-x^2/2\xi^2)\cos\Omega t$ and take the long time limit $t \gg \gamma$. (We have introduced two constants: ξ is a length, and Ω is a frequency.) Show that the asymptotic behaviour of the field is given by

$$\phi(x,t) = -\frac{\xi}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \frac{(\Omega^2 - c^2k^2)\cos\Omega t - 2\gamma\Omega\sin\Omega t}{(\Omega^2 - c^2k^2)^2 + 4\gamma^2\Omega^2} (\cos kx) e^{-\frac{1}{2}k^2\xi^2}.$$

6. Suppose that $\phi(x,t)|_{t=0} = p(x)$ and $\partial_t\phi(x,t)|_{t=0} = q(x)$ are known initial conditions and that the forcing function is also known and nonzero. Roughly sketch out how you would determine $\phi(x,t)$ for all future $t > 0$. (If you're feeling adventurous, you might want to select a specific form for each of $p(x)$ and $q(x)$ and plot up time snapshots of the field's time evolution in Mathematica. One straightforward choice is a plucked string: $p(x)$ gaussian and $q(x) = 0$.)