Physics 651: Exercise 1

(not for submission)

1. The natural numbers are *closed under addition* in the sense that any pair of natural numbers a and b has a sum $a + b$ that is also a natural number. Similarly, F is closed for multiplication if $\forall a, b \in F$ it follows that $a \cdot b = a \times b = ab \in F$. Which of the following fields is not closed under multiplication? Give a specific counterexample.

   (a) The whole numbers $\mathbb{N} \setminus \{0\}$
   (b) The natural numbers $\mathbb{N}$
   (c) The integers $\mathbb{Z}$
   (d) The rationals $\mathbb{Q}$
   (e) The irrationals $\mathbb{R} \setminus \mathbb{Q}$
   (f) The reals $\mathbb{R}$
   (g) The complex numbers $\mathbb{C}$

2. Given two arbitrary complex numbers $z_1 = x_1 + iy_1 = r_1 e^{i\theta_1}$ and $z_2 = x_2 + iy_2 = r_2 e^{i\theta_2}$, which one of the following is an <u>incorrect</u> statement?

   (a) $|z_1| = \sqrt{x_1^2 + y_1^2} = r_1$
   (b) $\bar{z}_1 z_2 = r_1 r_2 e^{i(\theta_2 - \theta_1)}$
   (c) $\bar{z}_1 + z_2 = (x_2 - x_1) + i(y_2 - y_1)$
   (d) $1/z_2 = \bar{z}_2 / |z_2|^2$

3. Given two numbers $z_1 = x_1 + iy_1 = e^{i\theta_1}$ and $z_2 = x_2 + iy_2 = e^{i\theta_2}$ on the unit circle in the complex plane, which one of the following is an <u>incorrect</u> statement?

   (a) $|z_1| = \sqrt{x_1^2 + y_1^2} = |z_2| = \sqrt{x_2^2 + y_2^2} = 1$
   (b) $z_1 z_2 = e^{i(\theta_1 + \theta_2)}$
   (c) $z_1 / z_2 = e^{i(\theta_1 - \theta_2)}$
   (d) $1/z_1 + 1/z_2 = e^{-i(\theta_1 + \theta_2)}$

4. Given two Grassman numbers θ and η, which <u>two</u> of the following expressions are <u>incorrect</u>?

   (a) $\theta\eta = -\eta\theta$
   (b) $\theta^2 = \eta^2 = 0$
   (c) $(\theta + \eta)^2 = 2\theta\eta$
   (d) $\int d\theta\,(3 + 2\theta) = 2$
   (e) $(2\theta - 3\eta)^2 = 12\eta\theta$

5. Given vectors u_i and v_i, a matrix $M_{i,j}$, and tensors $S_{i,j,k}$ and $T_{i,j,k,l}$ (where each of the indices i, j, k, and l range from 1 to n), which one of the following is <u>not</u> a dimensionally compatible expression?

   (a) $u_i = \sum_{j,k} S_{j,k,l} M_{i,l}$
   (b) $M_{i,j} = (u \otimes v)_{i,j} = u_i v_j$
   (c) $S_{i,j,k} = \sum_l M_{i,k} T_{i,k,l,j} u_l$
   (d) $1 = \operatorname{tr} M^3 = \sum_{i,j,k} M_{i,j} M_{j,k} M_{k,i}$

6. $A_{i,j}$ represents the element in the ith row and jth column of the matrix A. (Similarly for B, C, and D.) Which one of the following index-notation expressions is equal to $\operatorname{tr} ABCD$?

(a) $A_{i,j}B_{j,k}C_{k,l}D_{l,m}$
(b) $A_{i,j}B_{k,l}C_{j,k}D_{l,i}$
(c) $D_{i,l}C_{k,i}B_{j,k}A_{l,j}$
(d) $D_{l,k}C_{k,l}B_{j,i}A_{i,j}$

7. $A_{i,j}$ represents the element in the ith row and jth column of the matrix A. (Similarly for B, C, and D.) Which one of the following index-notation expressions is equal to $\operatorname{tr} C \operatorname{tr} BAD$?

   (a) $A_{i,j}B_{j,k}C_{k,l}D_{l,m}$
   (b) $A_{i,j}B_{k,l}C_{j,k}D_{l,i}$
   (c) $D_{j,k}C_{i,i}B_{l,j}A_{k,l}$
   (d) $D_{i,j}A_{l,i}C_{k,k}B_{j,l}$

8. Given vectors u_i and v_i and a matrix $M_{i,j}$—where all indices range from 1 to n and all entries are *complex-valued*—which of the following necessarily describes a *hermitian* matrix A?

   (a) $A_{i,j} = (u \otimes v)_{i,j} = u_i v_j$
   (b) $A_{i,j} = \frac{1}{2}(M_{i,j} + M^*_{j,i})$
   (c) $A_{i,j} = \frac{1}{2}(M_{i,j} + M_{j,i})$
   (d) $A_{i,j} = u_i + v_i + u_j + v_j$

9. The determinant of a product of square matrices A, B, C, D obeys which one of the following relations?

   (a) $\det ABCD = \det A + \det B + \det C + \det D$
   (b) $\det ABCD = \det A - \det B + \det C - \det D$
   (c) $\det ABCD = (\det A)(\det B)(\det C)(\det D)$
   (d) $\det ABCD = 1 + (\det A)\big[1 + (\det B)\big[1 + (\det C)\big[1 + (\det D)\big]\big]\big]$

10. Suppose that A, B, and C are square matrices of common dimension and that I is the corresponding identity matrix. The determinant of $A + BC$ is given by which of the following expressions?

   (a) $\det(A + BC) = \det A \det BC$
   (b) $\det(A + BC) = \det A \det(I + CA^{-1}B)$
   (c) $\det(A + BC) = \det A + \det BC$
   (d) $\det(A + BC) = \det(B^{-1}AC^{-1} + I)$

11. The trace of a product of square matrices A, B, C, D obeys which one of the following relations?

   (a) $\operatorname{tr} ABCD = \operatorname{tr} DABC$
   (b) $\operatorname{tr} ABCD = \operatorname{tr} DCBA$
   (c) $\operatorname{tr} ABCD = (\operatorname{tr} A)(\operatorname{tr} B)(\operatorname{tr} C)(\operatorname{tr} D)$
   (d) $\operatorname{tr} ABCD = \operatorname{tr} A + \operatorname{tr} B + \operatorname{tr} C + \operatorname{tr} D$

12. The trace of a product of square matrices A and B obeys which one of the following relations?

   (a) $\operatorname{tr} A^2B^3 = \operatorname{tr} BABAB$
   (b) $\operatorname{tr} A^2B^3 = \operatorname{tr} BBAAB$
   (c) $\operatorname{tr} A^2B^3 = \operatorname{tr} A^2 \operatorname{tr} B^3$
   (d) $\operatorname{tr} A^2B^3 = (\operatorname{tr} A)^2(\operatorname{tr} B)^3$
   (e) $\operatorname{tr} A^2B^3 = 2 \operatorname{tr} A + 3 \operatorname{tr} B$