Physics 651: Assignment 5

(to be submitted by Thursday, November 16, 2023)

To begin, we summarize the key results of vector calculus that we discussed in class. In our notation, f is a scalar field and $\mathbf{F}$ is a vector field; the nabla symbol denotes the gradient (viz., $\nabla = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$ in rectangular coordinates, $\nabla = \hat{\rho}\partial_\rho + (1/\rho)\hat{\phi}\partial_\phi + \hat{z}\partial_z$ in cylindrical polar); and ∂R represents the boundary of some region R.

- The *fundamental theorem for line integrals* states that

  $$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}_2) - f(\mathbf{r}_1),$$

  where C is a directed contour from $\mathbf{r}_1$ to $\mathbf{r}_2$. The result depends only on the starting and end points and not on the particular path taken by C. (It explains why forces derived from a potential must obey an energy conservation law.)

- The *divergence theorem* (also known as Gauss's theorem or Ostrogradsky's theorem) states that

  $$\int_V \nabla \cdot \mathbf{F}\, dV = \int_{\partial V} \mathbf{F} \cdot d\mathbf{S},$$

  where dV is a volume element, and $d\mathbf{S} = \hat{n}dS$ is the directed surface element pointing to the exterior of V. This result connects the charges contained in V to the flux through its boundary surface.

- *Stoke's theorem* states that

  $$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

  Here, S is an open surface, and ∂S is a contour along the boundary of S directed in a right-hand sense with respect to the orientation of $d\mathbf{S}$. This result connects the circulation of a field on the surface to the field's net contribution around the surface's edge.

1. Consider the vector field

   $$\mathbf{F} = \mathbf{F}(\rho, \phi, z) = \frac{\rho\hat{\rho}\sin^2\phi + \rho(\cos\phi\sin\phi)\hat{\phi} + z\hat{z}}{(z^2 + \rho^2\sin^2\phi)^{3/2}},$$

   expressed in cylindrical polar coordinates.

   (a) Explain why the position $\mathbf{r} = \rho\hat{\rho} + z\hat{z}$ has a differential $d\mathbf{r} = (d\rho)\hat{\rho} + \rho(d\phi)\hat{\phi} + (dz)\hat{z}$.

   (b) Consider the line integral along a contour C that can be parameterized by $\rho(t) = \ell t$, $\phi(t) = \pi t/2$, $z(t) = \ell\cos\pi t$ with t ranging from 0 to 1. Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 dt\, \cdots$ explicitly to obtain $(2 - \sqrt{2})/2\ell$.

   (c) Find a scalar field $V(\mathbf{r})$ such that $\mathbf{F} = -\nabla V$.

   (d) Now use the fundamental theorem for line integrals to confirm that

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \frac{2 - \sqrt{2}}{2\ell}.$$

2. Verify Stokes' theorem for the vector field $\mathbf{F}(x, y, z) = -xy\hat{x} + x\hat{y} + xz\hat{z}$ and surface S, where S is a 2×2 square patch centred on the origin with corners at $(-1, -1, 0)$ and $(1, 1, 0)$.

3. Prove the following:

   (a) For 3-vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$,

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\epsilon_{i,j,k}a_i b_j c_k = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{c} = (\mathbf{b}\times\mathbf{c})\cdot\mathbf{a} = (\mathbf{c}\times\mathbf{a})\cdot\mathbf{b};$$

   (b) $\nabla\cdot(\nabla\times\mathbf{a}) = 0$;

   (c) $\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$.

4. The function $\pi^2 - x^2$ peaks at $x = 0$ and falls off to zero at $x = \pm\pi$. Its 2π-periodic extension can be constructed from a cosine Fourier series (with an infinite number of terms). The following code demonstrates the series cut off after just five terms.

```
bump[x_] = \[Pi]^2 - x^2
Plot[bump[x], {x, -\[Pi], \[Pi]}]
FourierCosCoefficient[bump[x], x, n]
bump5[x_] = FourierCosSeries[bump[x], x, 5]
Plot[{bump[x], bump[x - 2 \[Pi]], bump[x + 2 \[Pi]], bump5[x]}, {x, -2.1 \[Pi], 2.1 \[Pi]},
    Frame -> True, PlotRange -> {{-7, 7}, {-2, 11}}]
```

   (a) Use *Mathematica* to create a log-linear plot of the discrepancy (in absolute value) between $\pi^2 - x^2$ and its 50-term cosine series approximation. You should find that the errors are smaller than 10^{-4} near $x = 0$. Justify the shape of resulting curve; in particular, explain why the error is so much larger near $x = \pm\pi$?

   (b) Now consider $(\pi^2 - x^2)(1 + \cos^2 10x)$ on the interval $[-\pi, \pi]$. Show that its the 20-term Fourier expansion is radically more faithful than the 19-term expansion. Then explain why.

5. We can use the ceiling operation to create a sawtooth wave $\lceil x\rceil - x$ of height 1 and period 1. The stretched function of period 2π is $\lceil x/2\pi\rceil - x/2\pi$. We can observe the convergence of its Fourier series as follows:

```
sawtooth[x_] = Ceiling[x/(2 \[Pi])] - x/(2 \[Pi])
saw2th[x_] = Sum[f[n] Exp[I n x], {n, -\[Infinity], \[Infinity]}]
Plot[{sawtooth[x], saw2th[x]}, {x, -15, 15}]
f[n_] = FourierCoefficient[sawtooth[x], x, n]
Plot[{Sum[f[n] Exp[I n x], {n, -8, 8}], sawtooth[x]}, {x, -15, 15}]
Plot[{Sum[f[n] Exp[I n x], {n, -16, 16}], sawtooth[x]}, {x, -15, 15}]
Plot[{Sum[f[n] Exp[I n x], {n, -32, 32}], sawtooth[x]}, {x, -15, 15}]
```

Now consider the function $\lceil x/\pi\rceil - x/\pi$. This new function actually has period π, but if we treat it as having period 2π and perform the Fourier analysis as above, what changes? Comment on the properties of the sawtooth wave's Fourier series coefficient, as represented by `f[n]`. Note its parity, and its behaviour at $n = 0$ and for even and odd n.