Physics 651: Assignment 6

(to be submitted by Thursday, November 3, 2022)

1. Test the following series for convergence:

$$\text{(a)} \sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}, \quad \text{(b)} \sum_{n=1}^{\infty} \frac{n!}{20^n}, \quad \text{(c)} \sum_{n=1}^{\infty} \frac{1}{n(n+2)}, \quad \text{(d)} \sum_{n=2}^{\infty} \frac{1}{n \ln n}.$$

In each case, say whether the series converges and how you found out.

2. Four quantum dots are arranged at the corners of an $\ell \times \ell$ square centred on the origin at positions $\boldsymbol{r}_1 = (\ell/2)(-\hat{x} + \hat{y})$, $\boldsymbol{r}_2 = (\ell/2)(+\hat{x} + \hat{y})$, $\boldsymbol{r}_3 = (\ell/2)(-\hat{x} - \hat{y})$, and $\boldsymbol{r}_4 = (\ell/2)(+\hat{x} - \hat{y})$. Each dot can carry charge—in proportion to the number n_i of electrons resident on the dot at position $\boldsymbol{r}_i$—and hence contributes an electrostatic potential

$$\Delta\phi_i = \frac{-en_i}{4\pi\epsilon_0 |\boldsymbol{r} - \boldsymbol{r}_i|}.$$

The total electrostatic potential is

$$\phi_{\text{tot}} = \Delta\phi_1 + \Delta\phi_2 + \Delta\phi_3 + \Delta\phi_4 = \sum_{\xi=\pm} \sum_{\eta=\pm} \frac{-en(\xi,\eta)}{4\pi\epsilon_0 |\boldsymbol{r} - (\ell/2)(\xi\hat{x} + \eta\hat{y})|},$$

with $n_1 = n(-,+)$, $n_2 = n(+,+)$, $n_3 = (-,-)$, and $n_4 = n(+,-)$. Express the total electrostatic potential as a power series in $1/r = 1/|\boldsymbol{r}|$. This should lead you to the multipole exansion

$$\phi_{\text{tot}} = \frac{q}{r} + \frac{\boldsymbol{r} \cdot \boldsymbol{P}}{r^3} + \frac{\boldsymbol{r} \cdot \overleftrightarrow{Q} \cdot \boldsymbol{r}}{2r^5} + \cdots, \quad \text{or equivalently,} \quad \phi_{\text{tot}} = \frac{q}{r} + \frac{\hat{r}^a P^a}{r^2} + \frac{\hat{r}^a Q^{a,b} \hat{r}^b}{2r^3} + \cdots,$$

written in terms of the total charge $q = -e \sum_{i=1}^4 n_i$, the dipole moment $P^a = -e \sum_{i=1}^4 r_i^a n_i$, and the quadrupole moment $Q^{a,b} = -e \sum_{i=1}^4 (3r_i^a r_i^b n_i - r_i^2 \delta^{a,b})$.

This *Mathematica* code snippet may be helpful:

```
r[1] = (a/2){-1, +1, 0}
r[2] = (a/2){+1, +1, 0}
r[3] = (a/2){-1, -1, 0}
r[4] = (a/2){+1, -1, 0}
V = Sum[-e n[i]/Sqrt[({x/b, y/b, z/b} - r[i]).({x/b, y/b, z/b} - r[i])], {i, 1, 4}]
Normal[Series[V, {b, 0, 3}]] /. b -> 1
```

3. There are some useful acceleration tricks that can transform a convergent series into one that converges to the same value but faster. Here we consider Aitken's method.

From the Taylor series expansion

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

define the partial sum

$$S_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1} x^k}{k}$$

and treat $(s_n) = (s_1, s_2, s_3, \ldots) = (S_1(1/2), S_2(1/2), S_3(1/2), \ldots)$ as a slowly converging sequence that approaches $\lim_{n\to\infty} s_n = \ln(1 + 1/2) = \ln(3/2)$.

Compute the terms of (s_n), the first dozen, say. Compare them graphically to the terms of an *accelerated* sequence defined by

$$s'_n = s_{n+2} - \frac{(s_{n+2} - s_{n+1})^2}{s_{n+2} - 2s_{n+1} + s_n}$$

I suggest that you prepare two plots, one showing s_n and s'_n for $n = 1, 2, \ldots, 10$ and another showing $\log_{10}|s_n - \ln(3/2)|$ and $\log_{10}|s'_n - \ln(3/2)|$. The second plot gives an estimate of the number of converged decimal digits as a function of n.