Physics 651: Assignment 4

(to be submitted by Tuesday, October 18, 2022)

To begin, we summarize the key results of vector calculus that we discussed in class. In our notation, f is a scalar field and $\boldsymbol{F}$ is a vector field; the nabla symbol denotes the gradient (viz., $\nabla = \hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z$ in rectangular coordinates, $\nabla = \hat{\rho}\partial_\rho + (1/\rho)\hat{\phi}\partial_\phi + \hat{z}\partial_z$ in cylindrical polar); and ∂R represents the boundary of some region R.

- The *fundamental theorem for line integrals* states that

$$\int_C \nabla f \cdot d\boldsymbol{r} = f(\boldsymbol{r}_2) - f(\boldsymbol{r}_1),$$

  where C is a directed contour from $\boldsymbol{r}_1$ to $\boldsymbol{r}_2$. The result depends only on the starting and end points and not on the particular path taken by C. (It explains why forces derived from a potential must obey an energy conservation law.)

- The *divergence theorem* (also known as Gauss's theorem or Ostrogradsky's theorem) states that

$$\int_V \nabla \cdot \boldsymbol{F}\, dV = \int_{\partial V} \boldsymbol{F} \cdot d\boldsymbol{S},$$

  where dV is a volume element, and $d\boldsymbol{S} = \hat{n}dS$ is the directed surface element pointing to the exterior of V. This result connects the charges contained in V to the flux through its boundary surface.

- *Stoke's theorem* states that

$$\int_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \int_{\partial S} \boldsymbol{F} \cdot d\boldsymbol{r}.$$

  Here, S is an open surface, and ∂S is a contour along the boundary of S directed in a right-hand sense with respect to the orientation of $d\boldsymbol{S}$. This result connects the circulation of a field on the surface to the field's net contribution around the surface's edge.

1. Consider the vector field

$$\boldsymbol{F} = \boldsymbol{F}(\rho, \phi, z) = \frac{2[\rho\hat{\rho} + (\ell^2/\rho)(\cos\phi)(\sin\phi)\hat{\phi} + z\hat{z}]}{(\rho^2 + \ell^2\sin^2\phi + z^2)^2},$$

   expressed in cylindrical polar coordinates.

   (a) Explain why the position $\boldsymbol{r} = \rho\hat{\rho} + z\hat{z}$ has a differential $d\boldsymbol{r} = (d\rho)\hat{\rho} + \rho(d\phi)\hat{\phi} + (dz)\hat{z}$.

   (b) Consider the line integral along a contour C that can be parameterized by $\rho(t) = \ell t$, $\phi(t) = \pi t/2$, $z(t) = \ell\cos\pi t$ with t ranging from 0 to 1. Substitute this specific contour parameterization into

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_C \frac{2[\rho d\rho + \ell^2(\cos\phi)(\sin\phi)d\phi + zdz]}{(\rho^2 + \ell^2\sin^2\phi + z^2)^2}.$$

   Evaluate the integral to obtain $2/3\ell^2$.

   (c) Find a scalar field $V(\boldsymbol{r})$ such that $\boldsymbol{F} = -\nabla V$.

   (d) Now use the fundamental theorem for line integrals to show that

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \frac{2}{3\ell^2}.$$

2. A potential $\phi(r) = Q/4\pi r^{1+\epsilon}$ gives rise to a field $\boldsymbol{E} = -\nabla\phi(r)$.

   (a) Provide an explicit expression for $\boldsymbol{E}(\boldsymbol{r})$.

   (b) Let V be a sphere of radius R centred on the coordinate origin. Compute $\int_{\partial V} \boldsymbol{E} \cdot d\boldsymbol{S}$ with an eye to its dependence on R.

   (c) Apply the divergence theorem, and argue that $Q\delta(\boldsymbol{r})$ can be interpreted as a point charge at the origin iff $\epsilon = 0$.

3. Verify Stokes' theorem for the vector field $\boldsymbol{F}(x, y, z) = -y\hat{x} + x\hat{y} - z\hat{z}$ and surface S, where S is a 2×2 square patch centred on the origin with corners at $(-1, -1, 0)$ and $(1, 1, 0)$.

4. Prove the following:

   (a) For 3-vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$,

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3} \epsilon_{i,j,k} a_i b_j c_k = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b};$$

   (b) $\nabla \cdot (\nabla \times \boldsymbol{a}) = 0$;

   (c) $\nabla \times (\nabla \times \boldsymbol{a}) = \nabla(\nabla \cdot \boldsymbol{a}) - \nabla^2 \boldsymbol{a}$.