Physics 651: Assignment 2

(to be submitted by Tuesday, September 13, 2022)

1. Suppose there are N particles (confined to one spatial dimension) with masses m_i and velocities v_i. Use the Cauchy-Swartz inequality to establish the following lower bound on the total kinetic energy:

$$E_{\text{kin}} \geq \frac{\left(\sum_{i=1}^{N} \sqrt{m_i/2}\right)^2}{\sum_{i=1}^{N} v_i^{-2}}.$$

2. A tiny ball of mass m rolls on a curved two-dimensional surface, parameterized by $z = x^2 - xy + 2y^2$. The xy-plane is horizontal, and gravity pulls in the $-\hat{z}$ direction. Hence, the gravitational potential energy is $U(x, y) = x^2 - xy + 2y^2$ (with x and y measured in metres and U measured in units of mg). Suppose that other mysterious forces are at work on the ball, such that its trajectory is confined to the closed curve $2x^2 + xy + 2y^2 = 1$. We would like to find the two points of locally minimum gravitational potential energy (consistent with the constraint) at which the ball could come to rest.

   (a) Let's group the horizontal coordinates into a row vector $r^T = (x\ y)$ and reexpress the gravitational potential energy as $U(r) = r^T A r$. What are the elements of the matrix A?

   (b) Now define a function

   $$L(r, \lambda) = r^T A r - \lambda\big(r^T B r - 1\big)$$

   in which the constraint has been added as a Lagrange multiplier term. What are the elements of B?

   (c) Show explicitly that $\nabla L = \partial L/\partial\lambda = 0$ leads to the generalized eigenvalue problem $Ar = \lambda Br$.

   (d) Find the two independent eigenvalue/eigenvector pairs, $(\lambda^{(1)}, r^{(1)})$ and $(\lambda^{(2)}, r^{(2)})$. Be sure that the $r^{(k)}$ vectors have been properly rescaled to satisfy the constraint.

   (e) Show that the gravitational energy values at the two locally stable points are $7/5$ and $1/3$.

   You may want to check your answer against the results of this Mathematica code listing:

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ContourPlot[x^2 - x y + 2 y^2, {x, -2, 2}, {y, -1, 1}]
ContourPlot[2 x^2 + x y + 2 y^2 == 1, {x, -2, 2}, {y, -1, 1}]
A = {{1, -1/2}, {-1/2, 2}}
B = {{2, 1/2}, {1/2, 2}}
sol = Eigensystem[{A, B}]
Eigensystem[Inverse[B].A] == sol
eval = First[sol]
evec = Last[sol]
n1 = evec[[1]].B.evec[[1]]
n2 = evec[[2]].B.evec[[2]]
ev1 = evec[[1]]/Sqrt[n1]
ev2 = evec[[2]]/Sqrt[n2]
ev1.B.ev1
ev2.B.ev2
ev1.A.ev1
ev2.A.ev2
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3. A particular quantum Hamiltonian has the following matrix representation, which involves two real-valued elements, ε_1 and ε_2, and one complex-valued element, Δ, all having units of energy:

$$H = \begin{pmatrix} \varepsilon_1 & \Delta^* \\ \Delta & \varepsilon_2 \end{pmatrix}.$$

(a) Using the notation $\Delta_1 = \operatorname{Re}\Delta$ and $\Delta_2 = \operatorname{Im}\Delta$, rewrite the Hamiltonian in terms of the 2×2 identity matrix, I, and the three Pauli matrices, σ_x, σ_y, and σ_z:

$$H = \frac{1}{2}(\varepsilon_1 + \varepsilon_2)I + \frac{1}{2}(\varepsilon_1 - \varepsilon_2)\sigma_z + \Delta_1\sigma_x + \Delta_2\sigma_y.$$

(b) Show that quantum evolution operator associated with H has the form

$$e^{-itH/\hbar} = e^{-i\omega_1 t}\left[I\cos\omega_2 t - \frac{it}{\hbar}\left(\frac{1}{2}(\varepsilon_1 - \varepsilon_2)\sigma_z + \Delta_1\sigma_x + \Delta_2\sigma_y\right)\operatorname{sinc}\omega_2 t\right],$$

where the angular frequencies are defined as

$$\omega_1 = (\varepsilon_1 + \varepsilon_2)/2\hbar \quad \text{and} \quad \omega_2 = \frac{1}{\hbar}\sqrt{\Delta_1^2 + \Delta_2^2 + \frac{1}{4}(\varepsilon_1 - \varepsilon_2)^2}.$$

(c) Consider the special case in which $\varepsilon_1 = \varepsilon_2$ (the two energy levels coincide) and $\Delta = \Delta_1 > 0$ is purely real and positive. (i) Show that if the system is prepared in the state

$$\psi(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

at time zero then the wave function at all subsequent times is given by

$$\psi(t) = e^{-itH/\hbar}\psi(0) = e^{-i\varepsilon_1 t/\hbar}\begin{pmatrix} \cos(t\Delta_1/\hbar) \\ -i\sin(t\Delta_1/\hbar) \end{pmatrix}.$$

(ii) Verify that $\psi^\dagger\psi = 1\ \forall t$ and that the probabilities of measuring the system in levels 1 and 2 are

$$p_1(t) = \psi^\dagger\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\psi = \cos^2\left(\frac{t\Delta_1}{\hbar}\right) \quad \text{and} \quad p_2(t) = \psi^\dagger\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\psi = \sin^2\left(\frac{t\Delta_1}{\hbar}\right).$$