Chaotic motion

Phys 750 Lecture 9

Finite-difference equations

- Finite difference equation approximates a differential equation as an iterative map

$$(x_{n+1}, v_{n+1}) = \mathcal{M}[(x_n, v_n)]$$

- Evolution from time $t = 0$ to $t_N = N \times \Delta t$ given by

$$(x_N, v_N) = \underbrace{\mathcal{M}[\mathcal{M}[\cdots \mathcal{M}}_{N \text{ times}}[(x_0, v_0)] \cdots]]$$

Finite-difference equations

- Backward analysis theorem: the iterative map is exactly solving some modified differential equation

$$\mathcal{M} \rightarrow v'(t) = \boxed{A(x(t), v(t), t)} + \epsilon$$

original PDE

- Are the additional ϵ terms "relevant"?
  - Can their effect be made arbitrarily small?
  - Do they change the physics qualitatively or break important conservation laws?

Hamiltonian systems

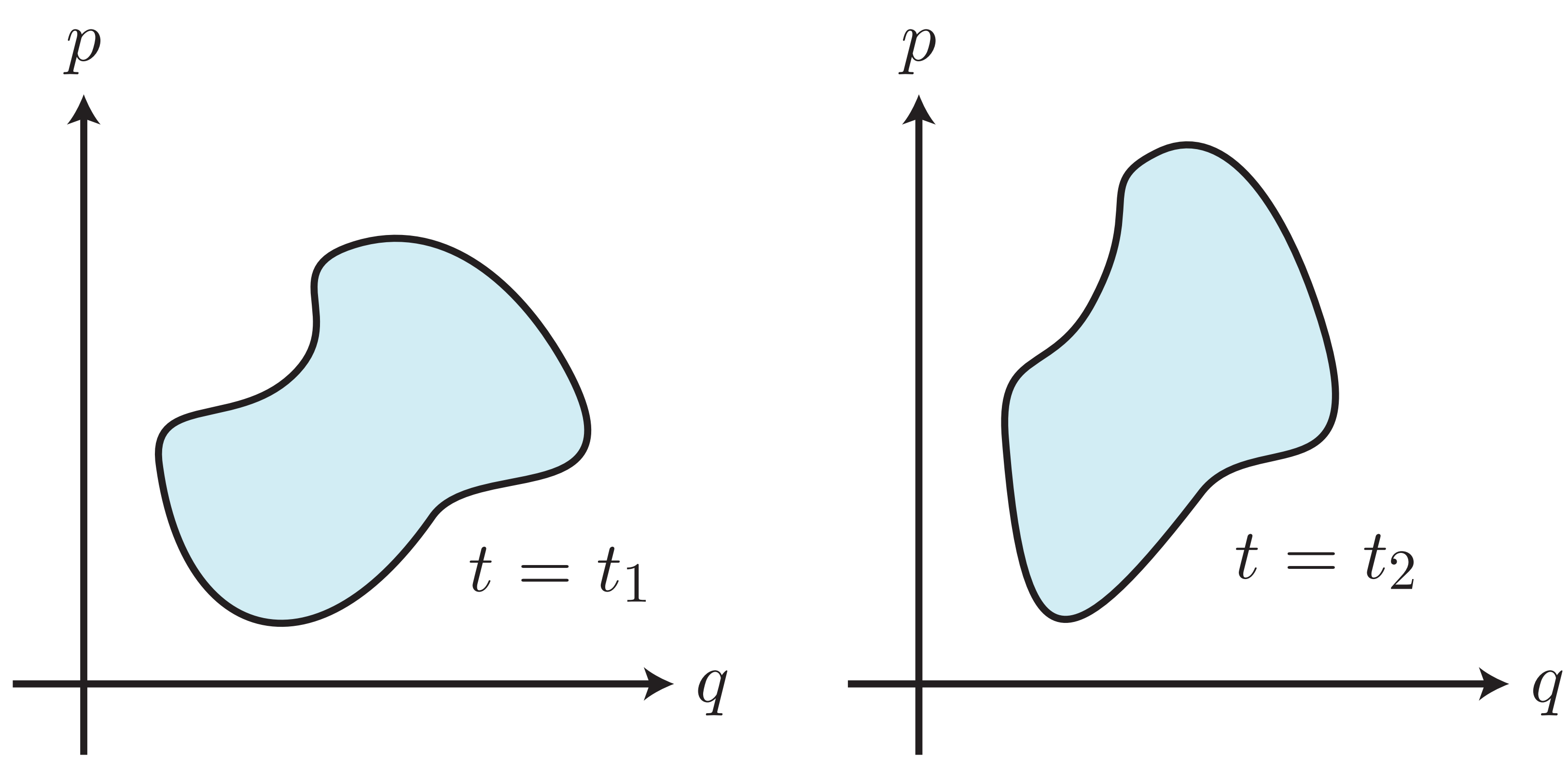
- Framework for non-dissipative, second-order ODEs
- System described by a Hamiltonian H in terms of generalized coordinates (p, q)
- Equations of motion:

$$\dot{p} \equiv \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

$$\dot{q} \equiv \frac{dq}{dt} = \frac{\partial H}{\partial p}$$

Hamiltonian systems

- Key property of Hamiltonian systems: symplectic symmetry = conservation of phase space volume



Limits to predictive power

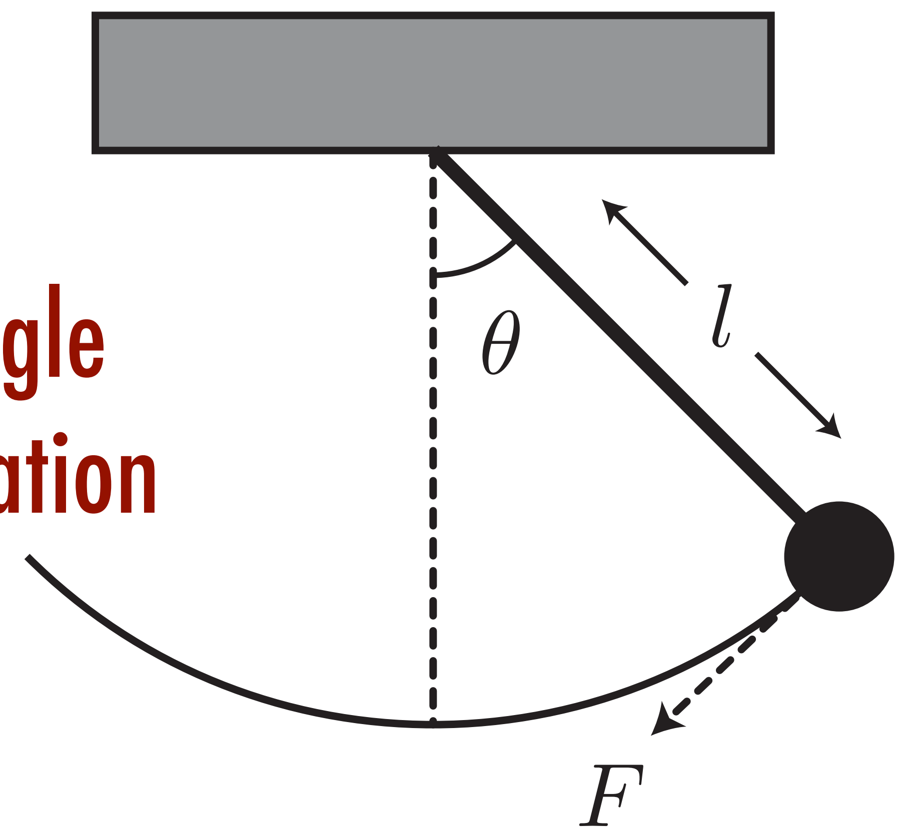
- Suppose that
  1. the finite difference scheme is high-order
  2. the resulting map $\mathcal{M}$ is well-behaved and correct up to irrelevant corrections
  3. the time step Δt is taken suitable small
- Are we then guaranteed a controlled, predicable solution? No.

Oscillatory motion

- Simple pendulum: linear "Hooke's Law" restoring force for small angular deviations

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta$$

small angle approximation



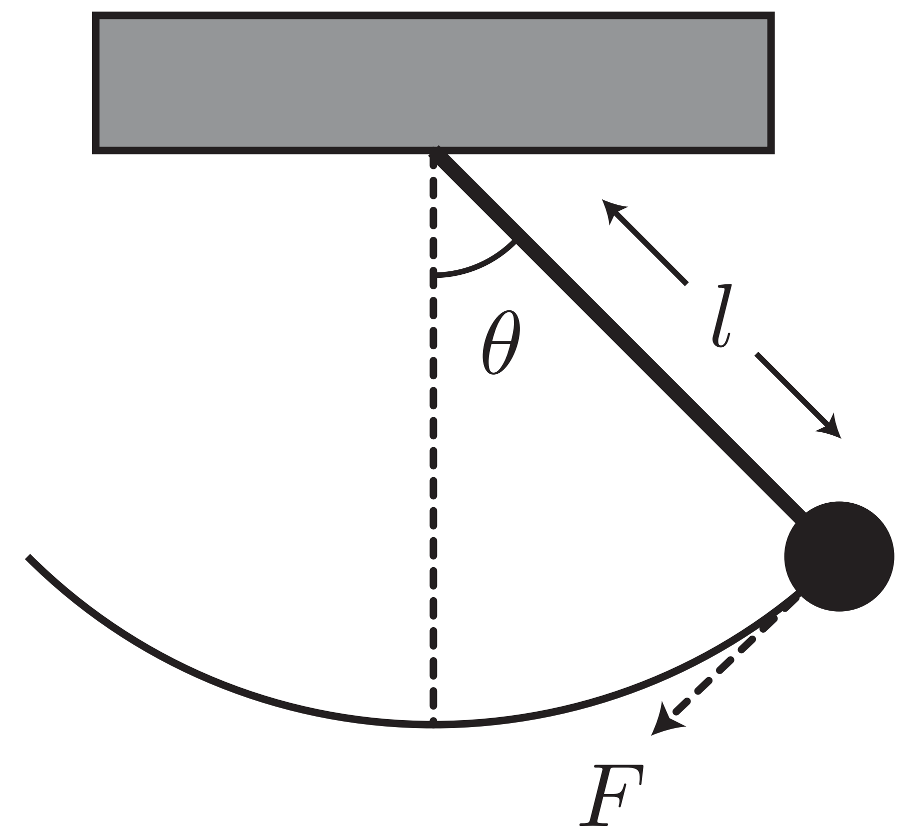
- Oscillatory solution

$$\theta(t) = \theta_0 \sin(\Omega t + \phi)$$

with characteristic angular frequency $\Omega = \sqrt{g/l}$

Oscillatory motion

- Important features:
  - oscillations are perfectly regular in time and continue forever (no decay)
  - angular frequency is independent of the mass m and amplitude θ_0



Oscillatory motion

- Usual trick for numerical solution: decompose the second-order ODE into a system of first-order equations

$$\frac{d\omega}{dt} = -\frac{g}{l}\theta$$
$$\frac{d\theta}{dt} = \omega$$

- Convert to a system of difference equations by discretizing the time variable, $t \to t_n = n \times \Delta t$

Oscillatory motion

- Simple pendulum is a Hamiltonian system described by an angular coordinate θ and angular momentum $L = I\omega$ where $I = ml^2$ is the moment of inertia:

$$H = \frac{L^2}{2I} + \frac{1}{2} I \Omega^2 \theta^2$$

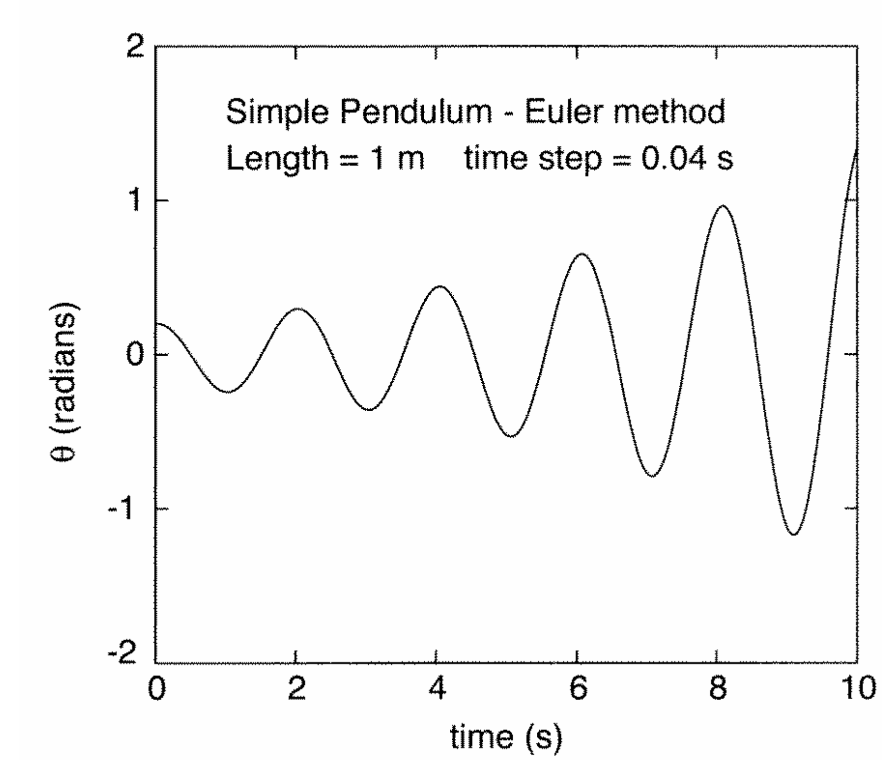
- Hamilton's equations reproduce the first-order pair:

$$\dot{L} = -\frac{\partial H}{\partial \theta} = -I\Omega^2\theta \qquad \frac{d\omega}{dt} = -\frac{g}{l}\theta$$

$$\longleftrightarrow$$

$$\dot{\theta} = \frac{\partial H}{\partial L} = \frac{L}{I} \qquad \frac{d\theta}{dt} = \omega$$

Oscillatory motion



- Hamiltonian system implies conservation of energy and preservation of the symplectic symmetry
- Neither are satisfied with Euler updates (small errors accumulate over each cycle)
- Important to use higher-order integration methods

Dissipation

- Simple pendulum is highly idealized
- More realistic models might include friction or damping terms proportional to the velocity

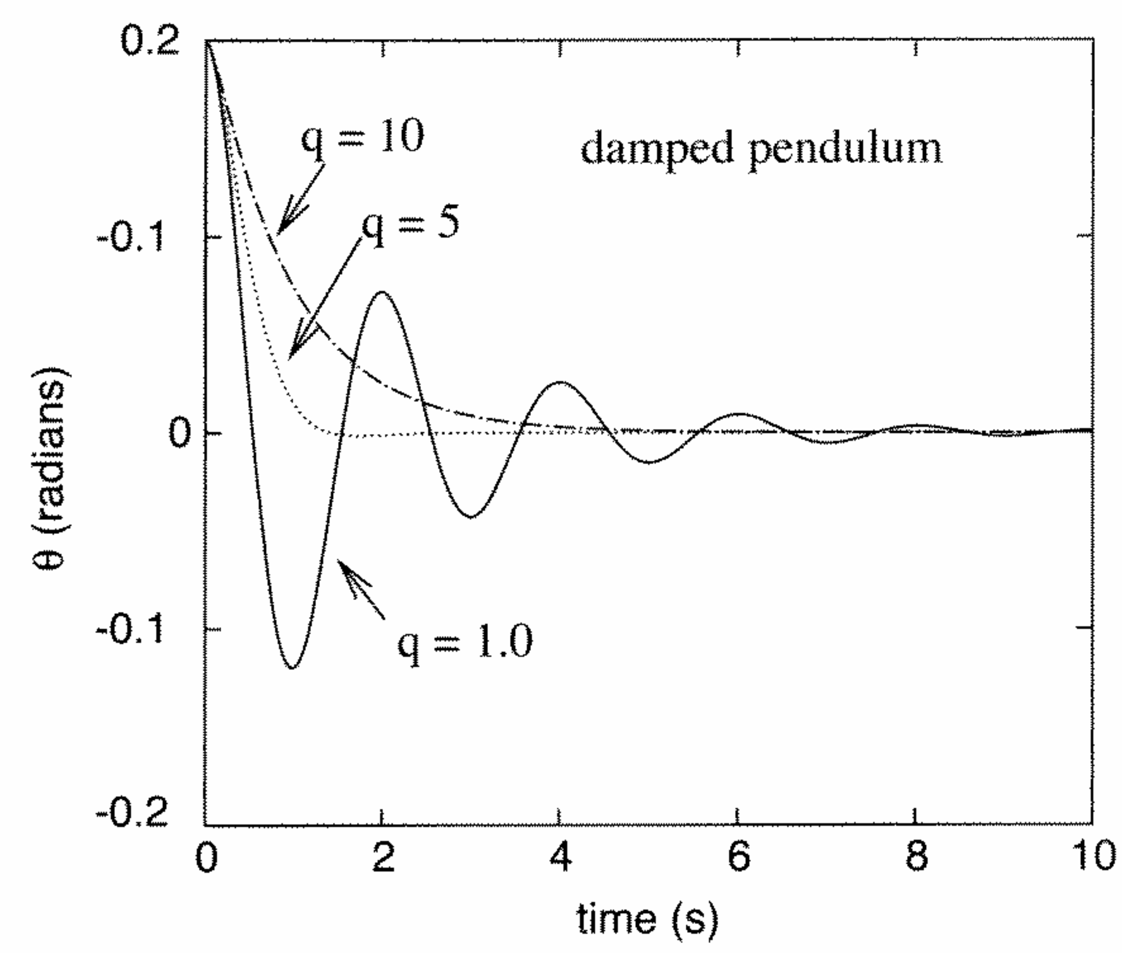
$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt}$$

- Boring: Analytic solution shows that oscillations die out

$$\theta(t) \sim e^{-qt/2}$$

Dissipation

- Depending on the strength of the parameter q, the behaviour may be under-, over-, or critically damped



$$\theta(t) = \theta_0 e^{-qt/2} \sin\left[\left(\Omega^2 - \tfrac{1}{4}q^2\right)^{1/2} t + \phi\right]$$

$$\theta(t) = \theta_0 \exp\left[-q/2 \pm \left(\tfrac{1}{4}q^2 - \Omega^2\right)^{1/2}\right] t$$

$$\theta(t) = \left(\theta_0 + Ct\right) e^{-qt/2}$$

Dissipation + driving force

- We now add a driving force to the problem
- Assume a sinusoidal form with strength F_D and angular frequency Ω_D

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - q\frac{d\theta}{dt} + F_D \sin \Omega_D t$$

- Pumps energy into or out of the system and competes with the natural frequency when $\Omega_D \neq \Omega$

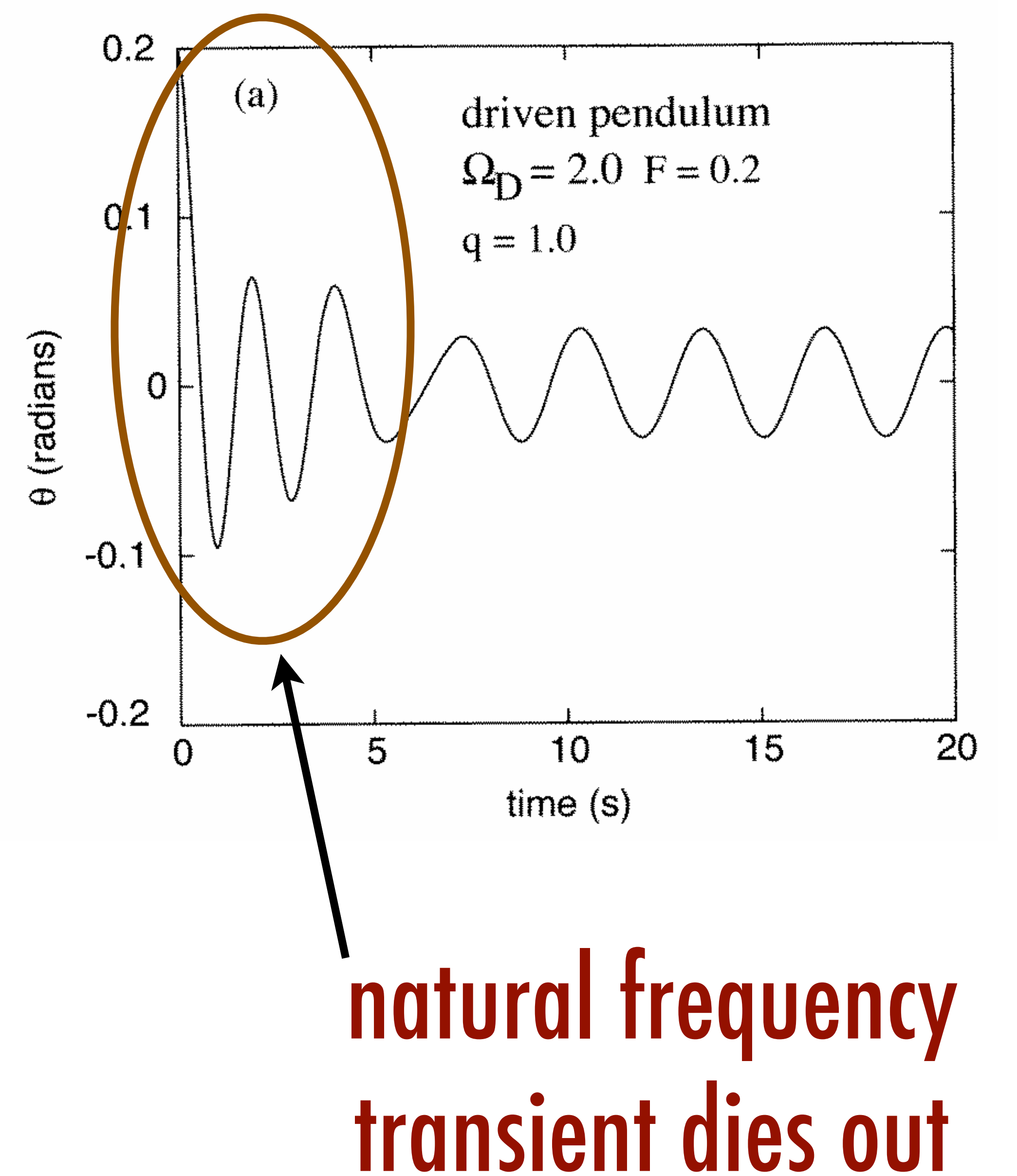
Dissipation + driving force

- In this case, the steady state solution is sinusoidal in the driving frequency:

$$\theta(t) = \theta_0 \sin(\Omega_D t + \phi)$$

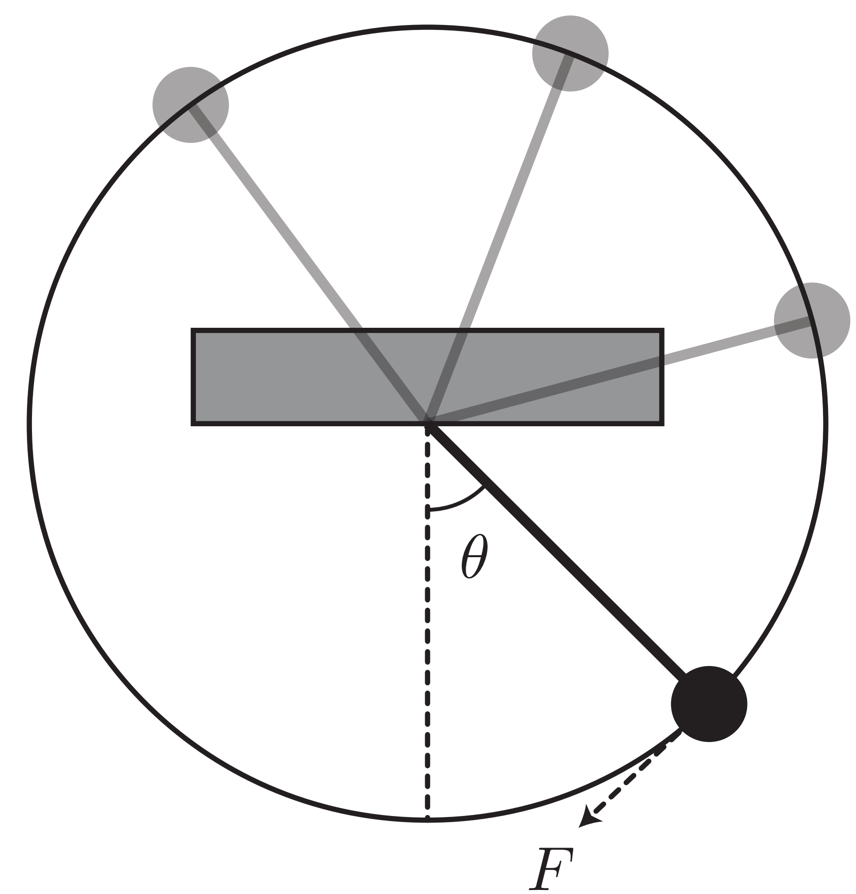
- But the amplitude has a resonance when the driving frequency is close to the natural frequency:

$$\theta_0 = \frac{F_D}{\sqrt{(\Omega^2 - \Omega_D^2)^2 + (q\Omega_D)^2}}$$



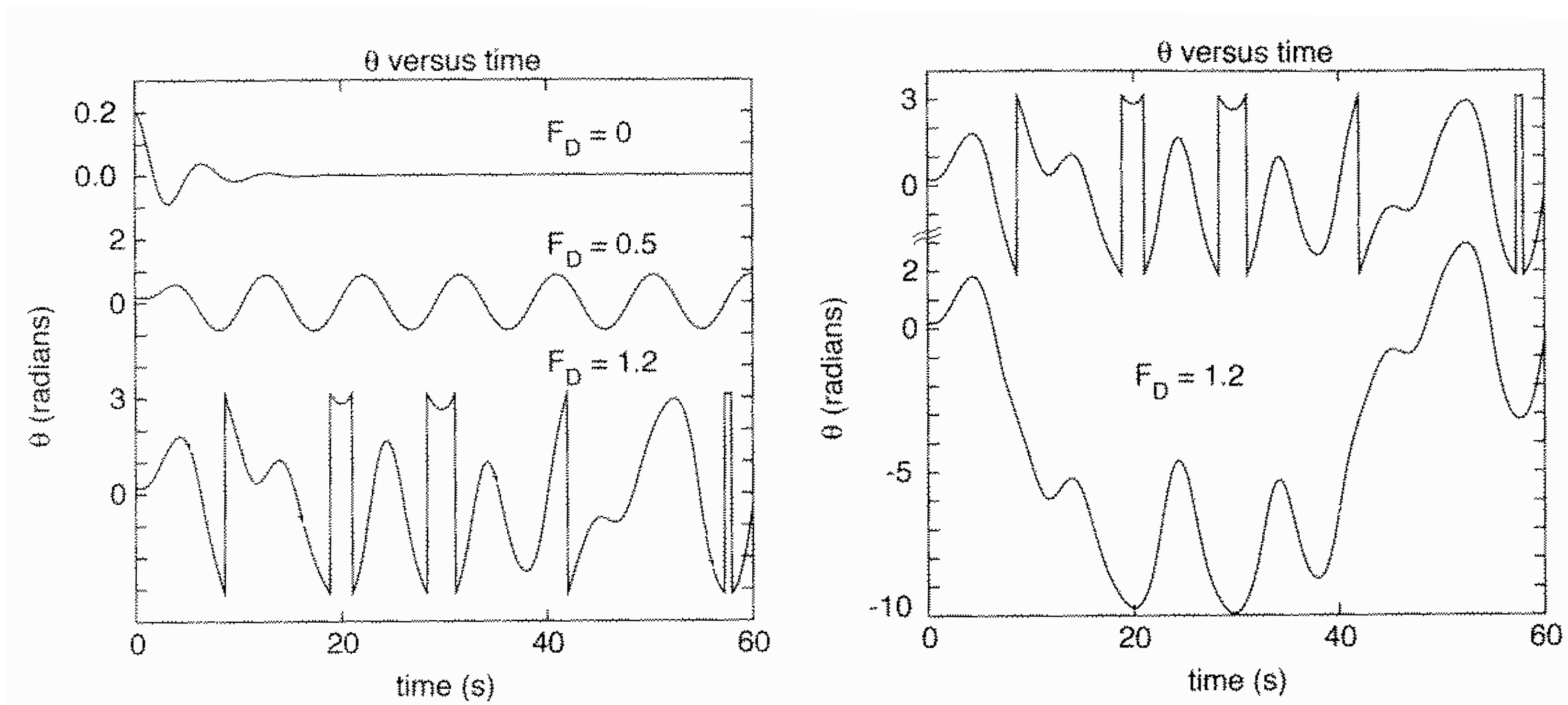
Non-linearity

- Small angular approximation is no longer appropriate
- Pendulum may swing completely around its pivot
- No analytical solution with a sinusoidal restoring force



$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta - q\frac{d\theta}{dt} + F_D \sin\Omega_D t$$

Chaotic motion



- When F_D is sufficiently large, the motion has no simple long-time behaviour
- The motion never repeats and is said to be chaotic
- But it is not random

Chaotic motion

- What does it mean to be non-repeating, unpredictable, and yet still deterministic?
- Remember: the behaviour is unique and governed by the specification of the initial value problem

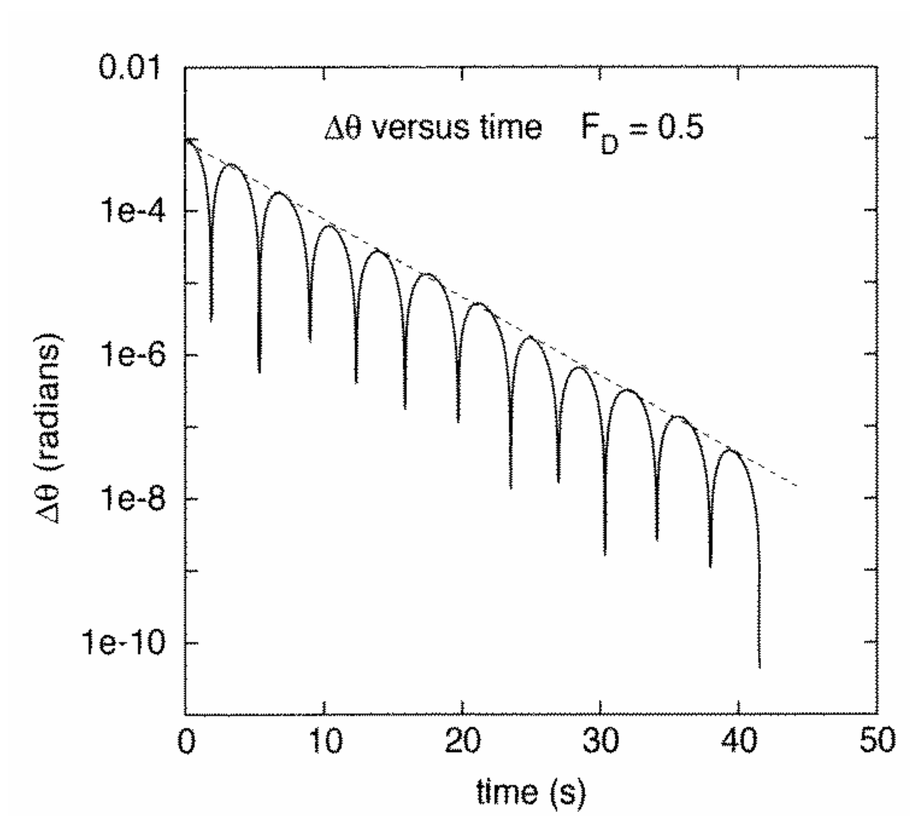
Chaotic motion

- For chaotic systems, infinitesimal variations in the initial conditions lead to different long-time behaviour
- For example, two identical chaotic pendulums with nearly identical initially conditions will show exponential growth in the angular distance $\Delta\theta \equiv |\theta_1 - \theta_2|$

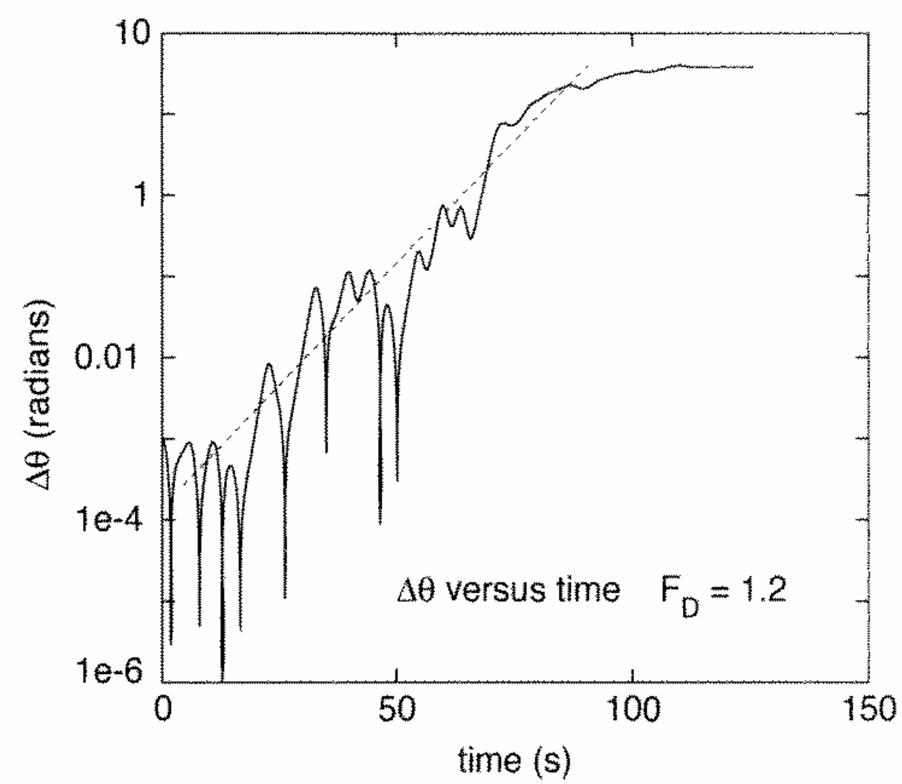
Transition to chaos

- Regular and chaotic regions distinguished by a change of sign of the Lyapunov exponent λ

$$\Delta\theta \sim e^{\lambda t}$$



$\lambda < 0$



$\lambda > 0$

Lyapunov exponents

- In general, there is a Lyapunov exponent associated with each phase-space degree of freedom

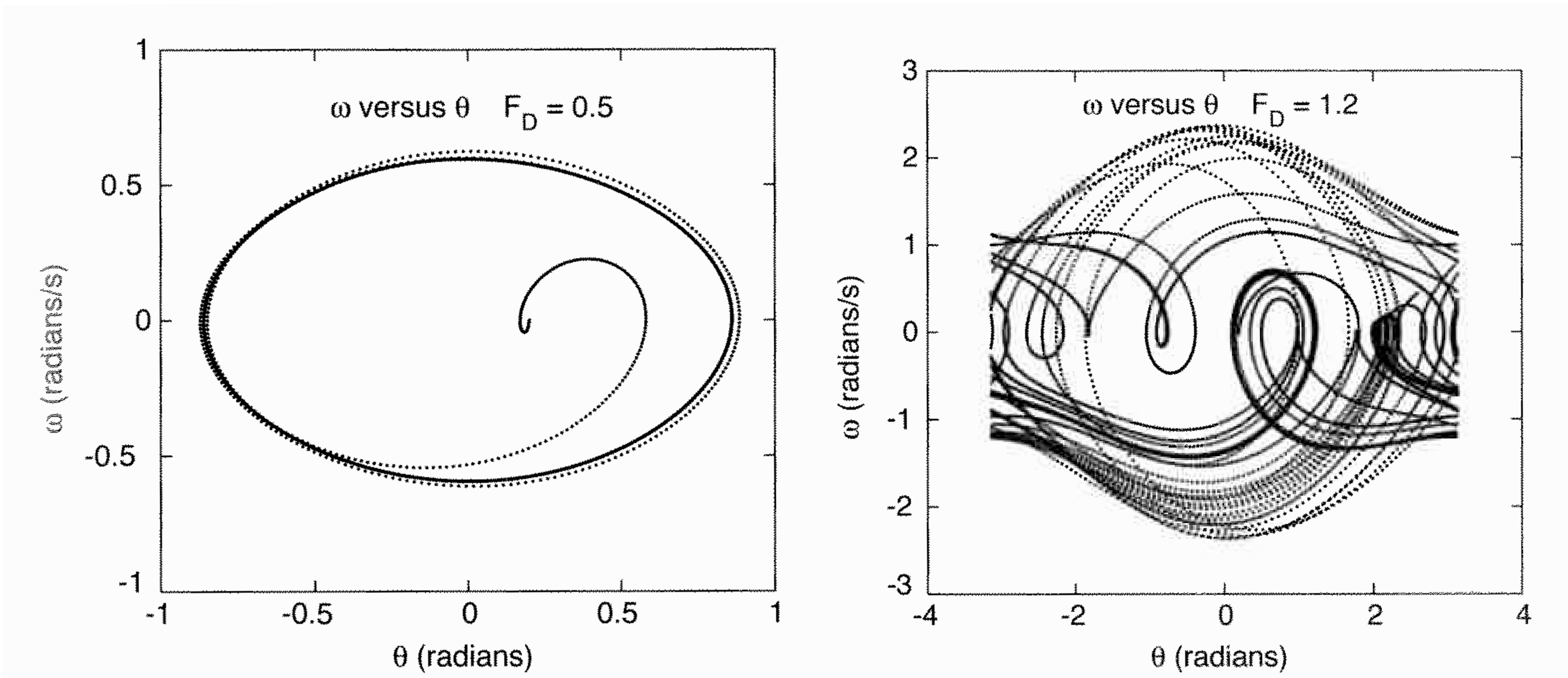
$$\begin{pmatrix} |\delta_1(t)| \\ |\delta_2(t)| \\ \vdots \\ |\delta_N(t)| \end{pmatrix} = M(t) \begin{pmatrix} |\delta_1(0)| \\ |\delta_2(0)| \\ \vdots \\ |\delta_N(0)| \end{pmatrix} \qquad M(t) = U^T \exp \begin{pmatrix} \lambda_1 t & & & \\ & \lambda_2 t & & \\ & & \ddots & \\ & & & \lambda_N t \end{pmatrix} U$$

- For a conservative system: $\sum_{k=1}^{N} \lambda_k = 0$

- For a dissipative system: $\sum_{k=1}^{N} \lambda_k < 0$

Unitary transformation: matrix of Eigenvectors of M

The view from phase space



- Chaotic path through phase space still exhibits structure
- There are many orbits that are nearly closed and persist for one or two cycles

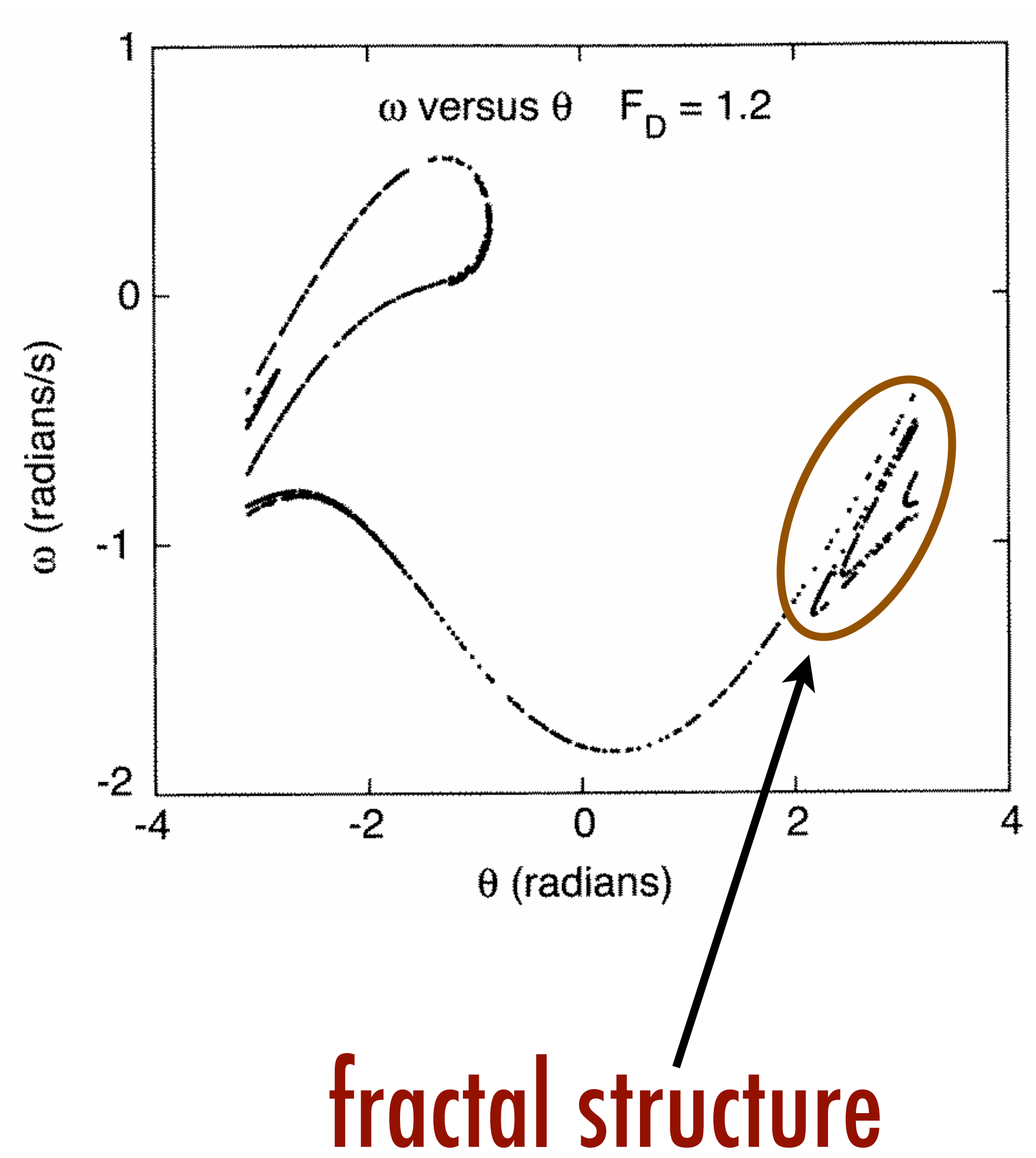
Strange attractor

- Poincaré section: only plot points at times in phase with the driving force

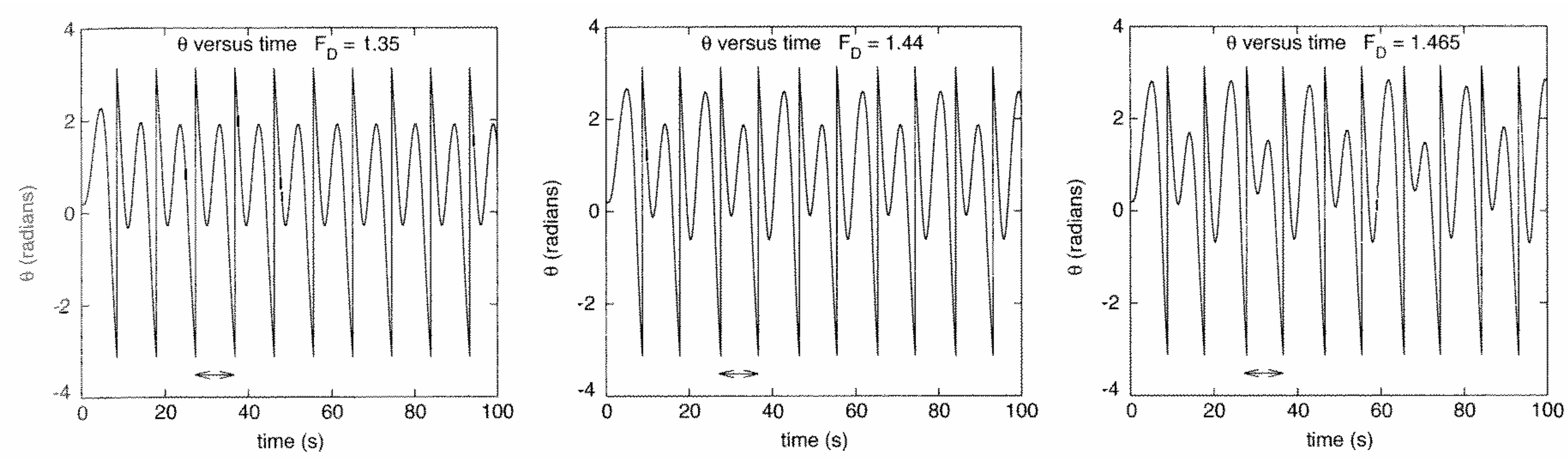
$$\Omega_D t = 2n\pi$$

for integer n

- For a wide range of initial conditions, trajectories lie on this surface of points, known as a strange attractor

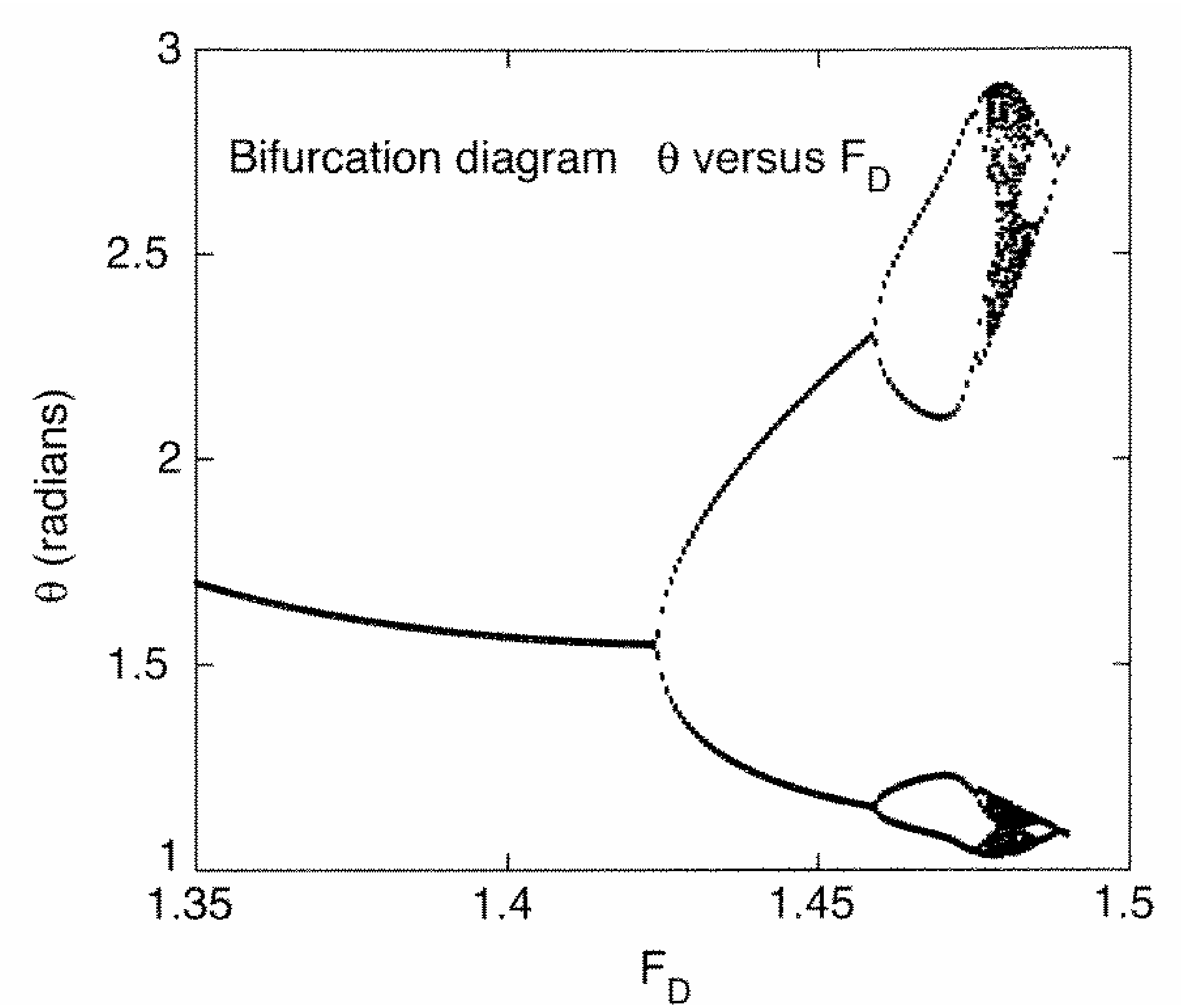


Period doubling



- For the pendulum, the route to chaos is via period doubling
- The system shows response at a subharmonic $\Omega_D/2$

Period doubling



- Bifurcation diagram plots a Poincaré section versus driving force
- Regularity in windows of period 2^n
- Feigenbaum delta:

$$\delta_n \equiv \frac{F_n - F_{n-1}}{F_{n+1} - F_n}$$

- Universal value $\delta \approx 4.669$