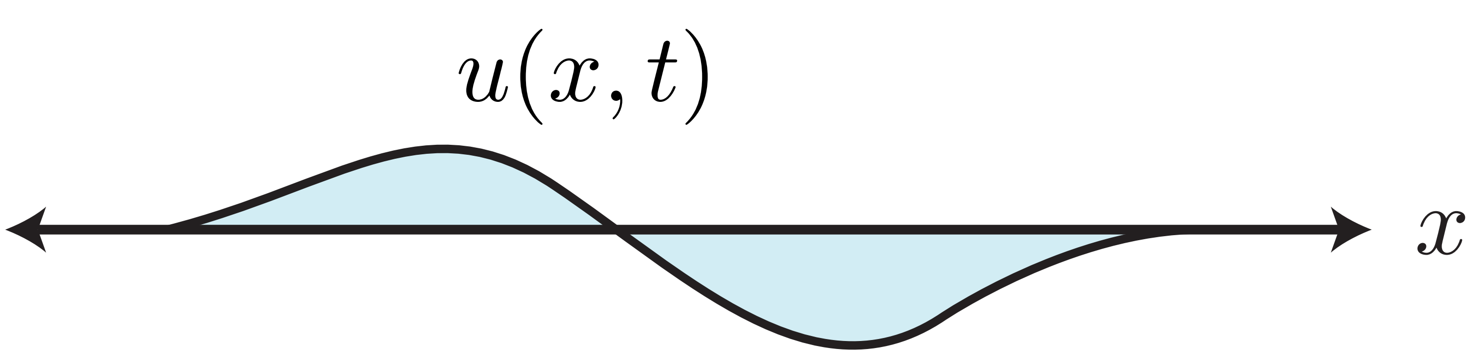
Wave motion and spectral methods

Phys 750 Lecture 13

Wave equation

- Wave motion is described by a partial differential equation in time and position
- For example: vibrations on a generic string described by a displacement field $u(x,t)$ evolving according to

$$\rho \frac{\partial^2 u}{\partial t^2} + R\frac{\partial u}{\partial t} = T\frac{\partial^2 u}{\partial x^2} - \kappa \frac{\partial^4 u}{\partial x^4} + \cdots$$



$\rho =$ mass per unit length

$R =$ loss coefficient

$T =$ line tension

$\kappa =$ nonlinear correction

Wave equation

- In the limit of small amplitudes, weak spatial modulation, and slow vibrations, the PDE is linear:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

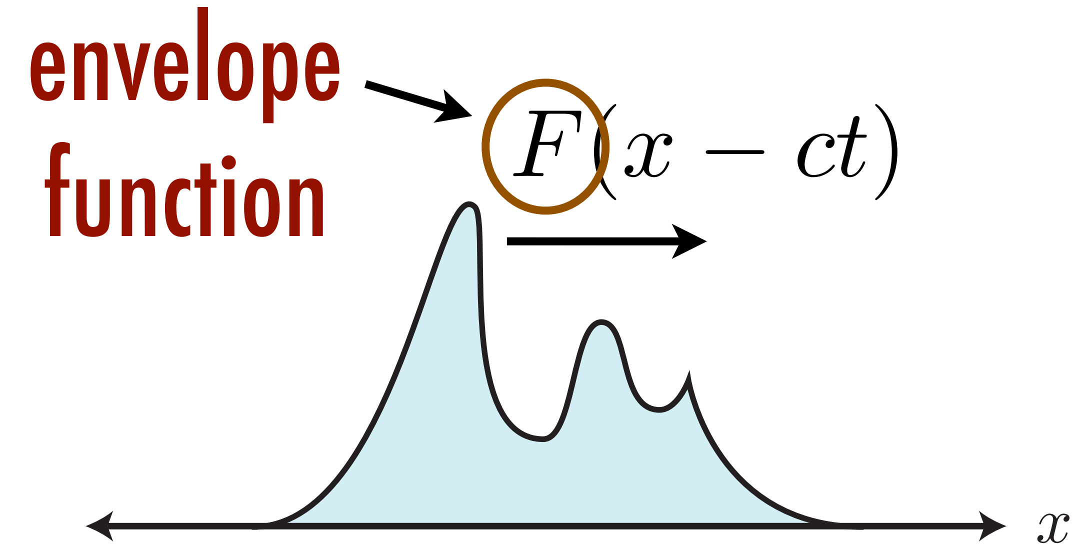
- The wave velocity $c = \sqrt{T/\rho}$ is a characteristic speed related to the string's tension and mass per unit length

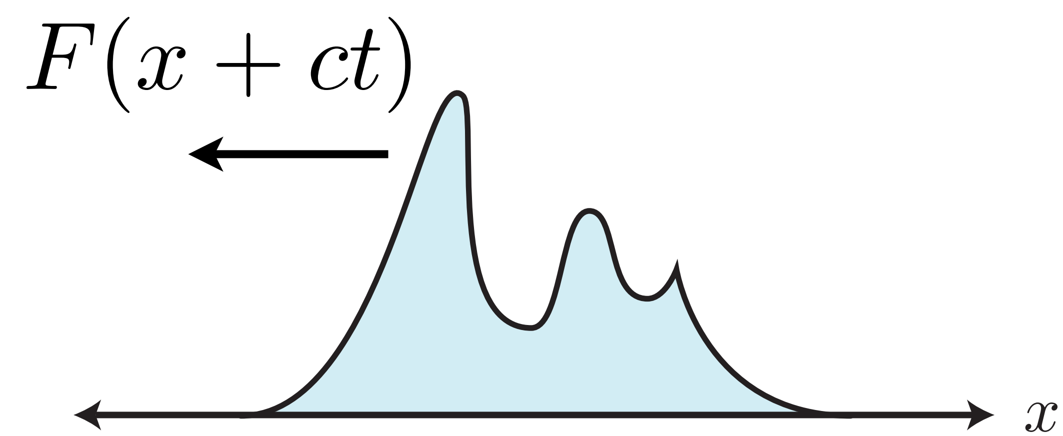
- Advection factoring:

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u(x,t) = 0$$

Wave equation

- For an infinite string, any right- or left-travelling wave is a solution
- Lossless propagation
- Since the PDE is linear, all linear superpositions of solutions are also solutions:





$$u(x, t) = aF(x \pm ct) + bG(x \pm ct) + \cdots$$

Space-time mesh

- Naïve approach is to apply usual discretization techniques
- PDE requires a double mesh: $u_i^{(n)} = u(i\Delta x, n\Delta t)$
- Numerical estimates of the time and space derivatives

$$\frac{u_i^{(n+1)} + u_i^{(n-1)} - 2u_i^{(n)}}{(\Delta t)^2} \approx c^2 \left[\frac{u_{i+1}^{(n)} + u_{i-1}^{(n)} - 2u_i^{(n)}}{(\Delta x)^2} \right]$$

lead to a recurrence relation

$$u_i^{(n+1)} = 2(1 - s^2)u_i^{(n)} - u_i^{(n-1)} + s^2\left(u_{i+1}^{(n)} + u_{i-1}^{(n)}\right)$$

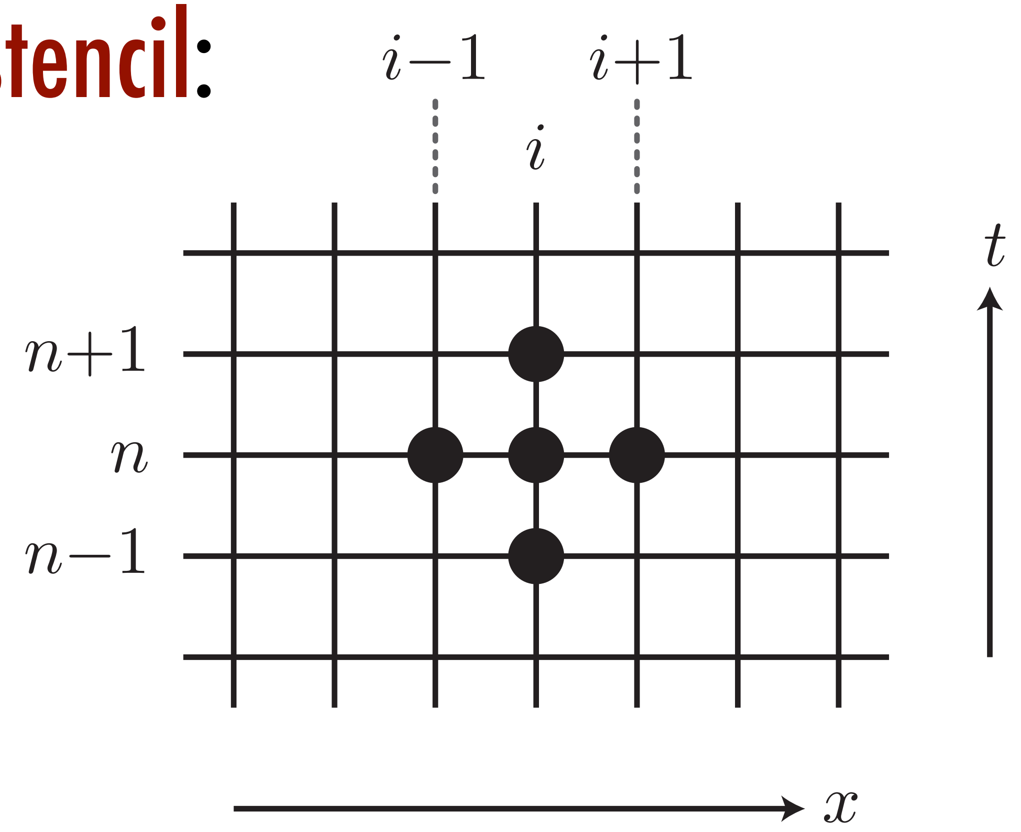
with $s = c\Delta t / \Delta x$ ⟵ controls the stability of the recursion

Space-time mesh

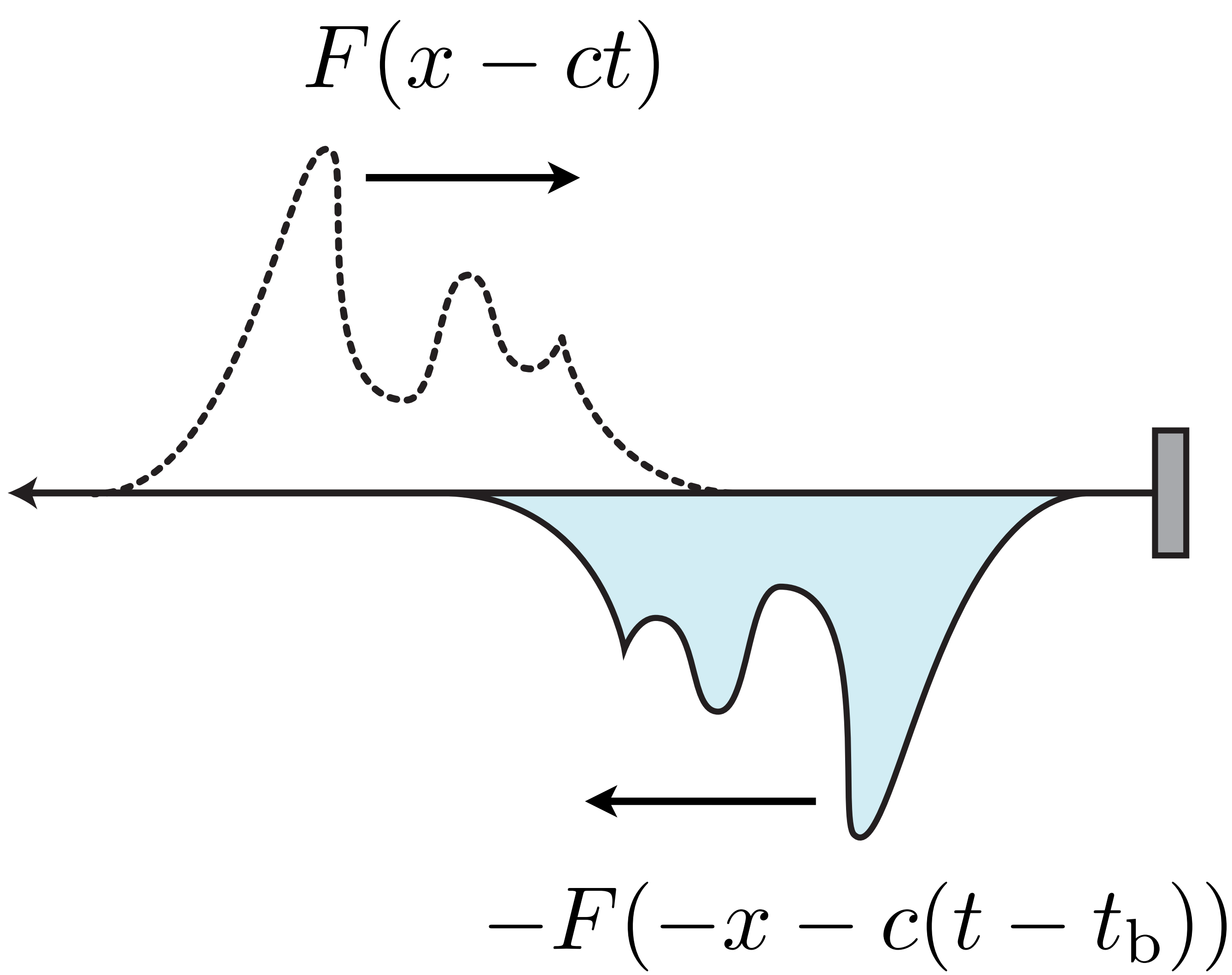
- How we assign the current field from its values at previous time steps – e.g.,

$$u_i^{(n+1)} = 2\left(1 - s^2\right)u_i^{(n)} - u_i^{(n-1)} + s^2\left(u_{i+1}^{(n)} + u_{i-1}^{(n)}\right)$$

– is represented by a stencil:

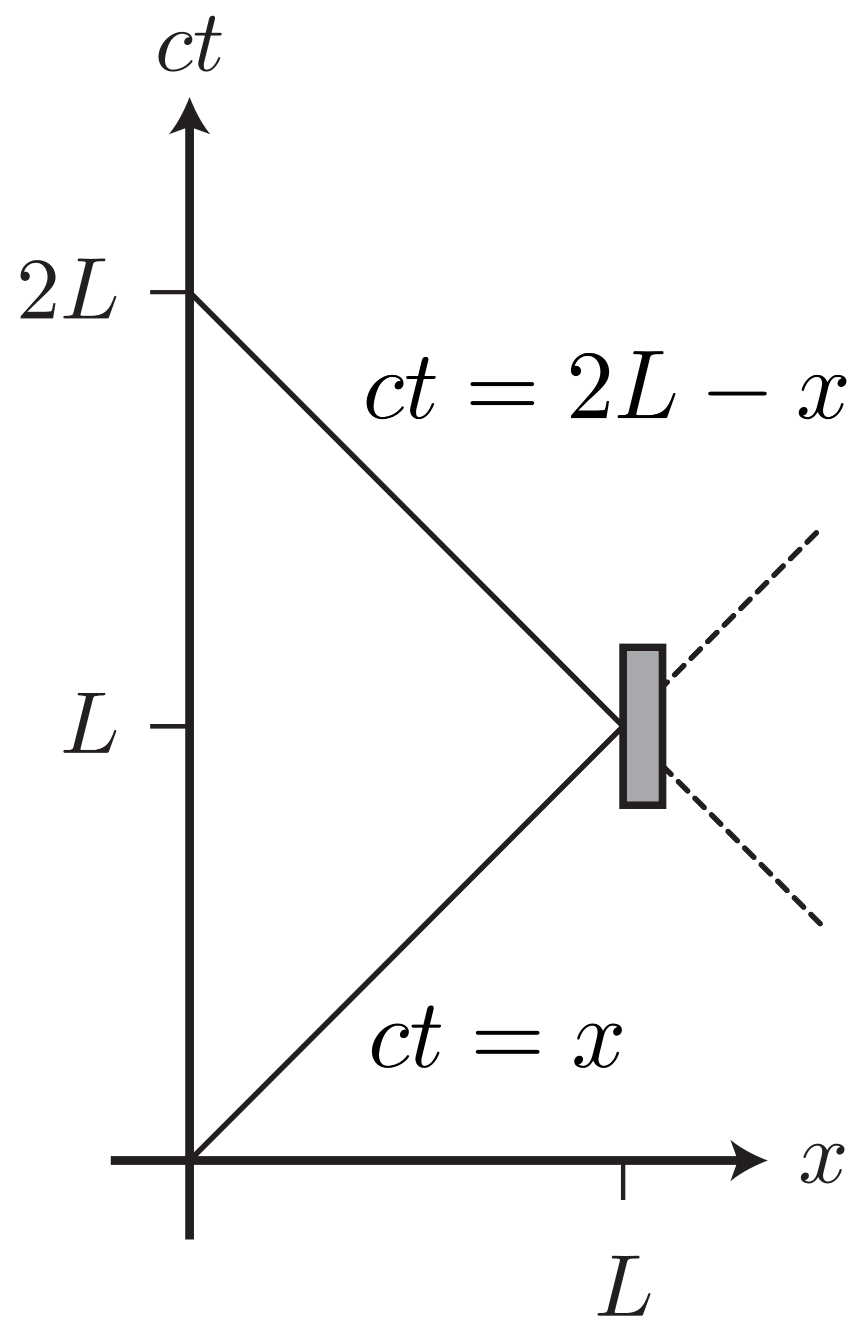


Reflections



- All finite systems have a boundary, where the wave is contained by reflection
- A fixed end leads to reversal of motion and a π phase shift
- Finite differences are troublesome there

Reflections

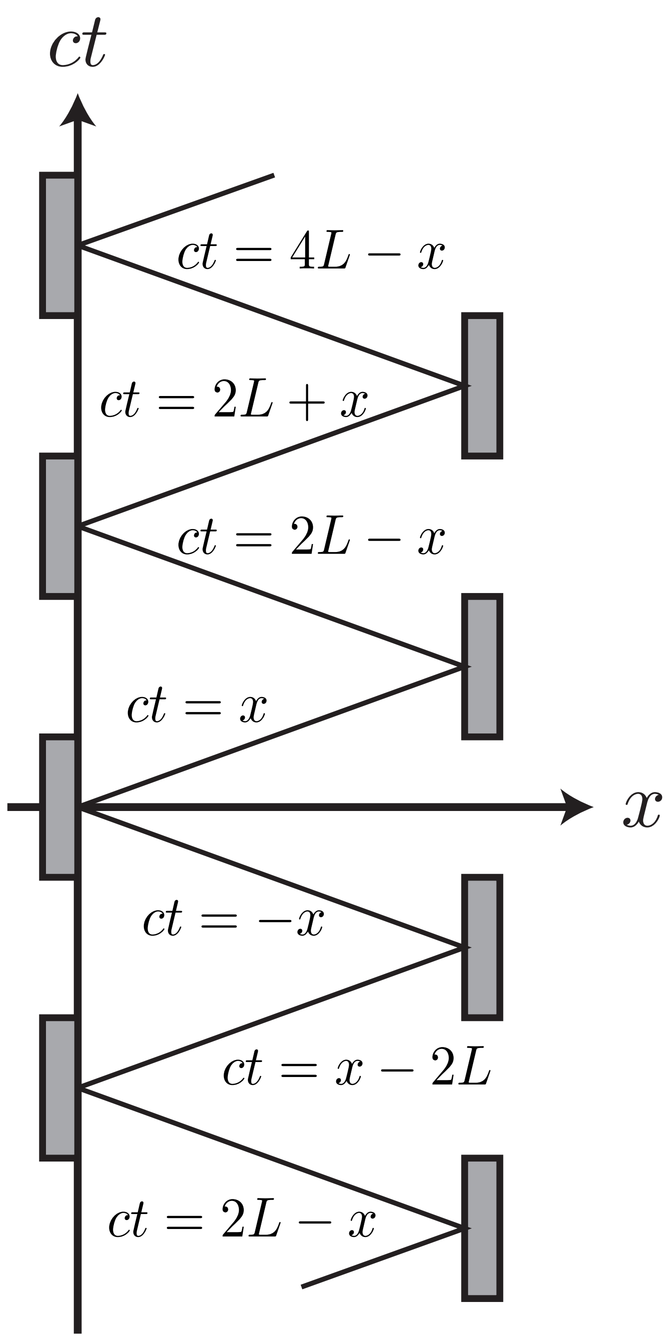


- Single reflection event leads to the superposed solution

$$u(x,t) = F(x - ct) - F(2L - x - ct)$$

- With the understanding that $u(x,t) = 0$ for $x > L$

Reflections



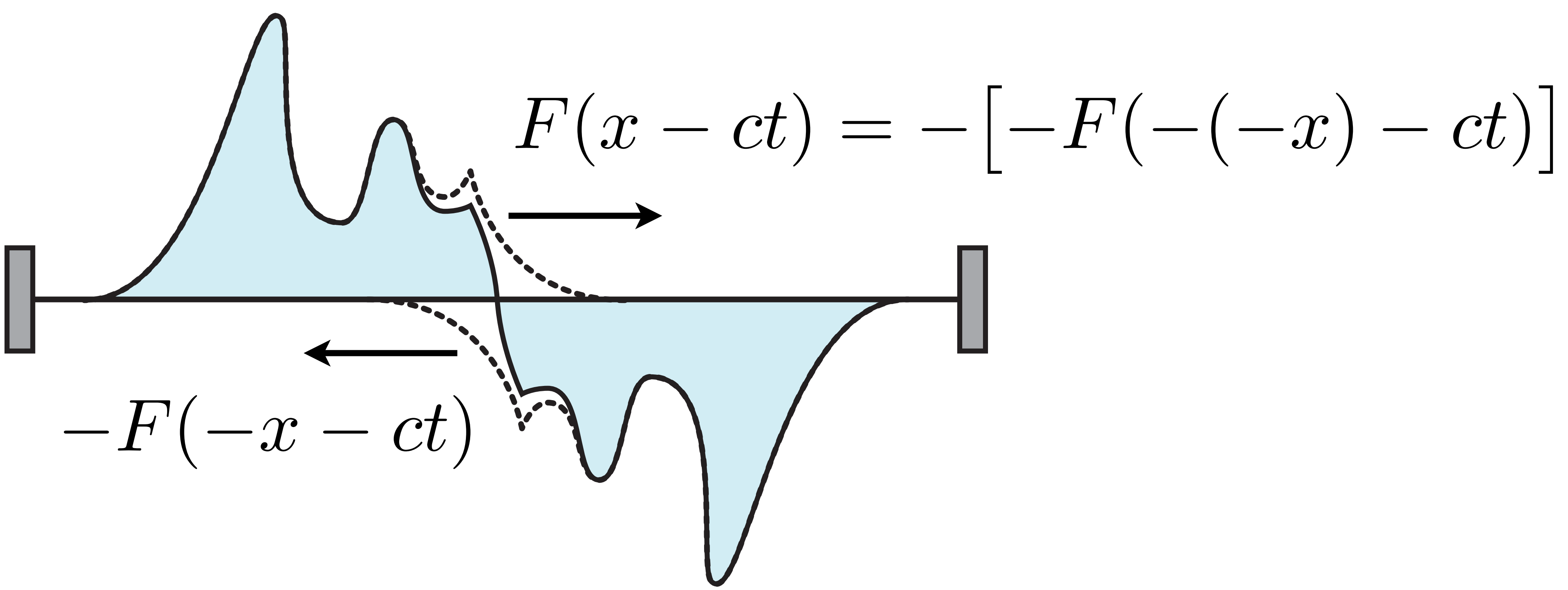
- String segment undergoes an infinite number of reflections:

$$u(x,t) = \sum_{n=-\infty}^{\infty} \Big[F(2nL + x - ct) - F(2nL - x - ct)\Big]$$

- Vanishes at $x = 0$ and $x = L$
- Periodic extension of the envelope function

Reflections

- A wave contained in a finite interval has a stable solution consisting of all multiply-reflected contributions



- Constructive parts: $u(x,t) \sim F(x - ct) - F(-x + ct)$

Normal modes

- Suppose that the wave is confined to $x \in [0, L]$
- Barring pathological examples, the periodic extension of F has a discrete Fourier representation

$$F(x) = \sum_{n=-\infty}^{\infty} F_n e^{in\pi x/L}$$

- The components F_n are arbitrary except that the resulting envelope function must be real ($F_n^* = F_{-n}$)

Normal modes

- Waves in motion have the form

$$F(\pm x - ct) = \sum_{n=-\infty}^{\infty} F_n e^{\pm i k_n x} e^{-i\omega_n t}$$

- Behaviour of each mode n is determined by a characteristic wave-vector and angular frequency

$$k_n = \frac{n\pi}{L},\ \omega_n = ck_n$$

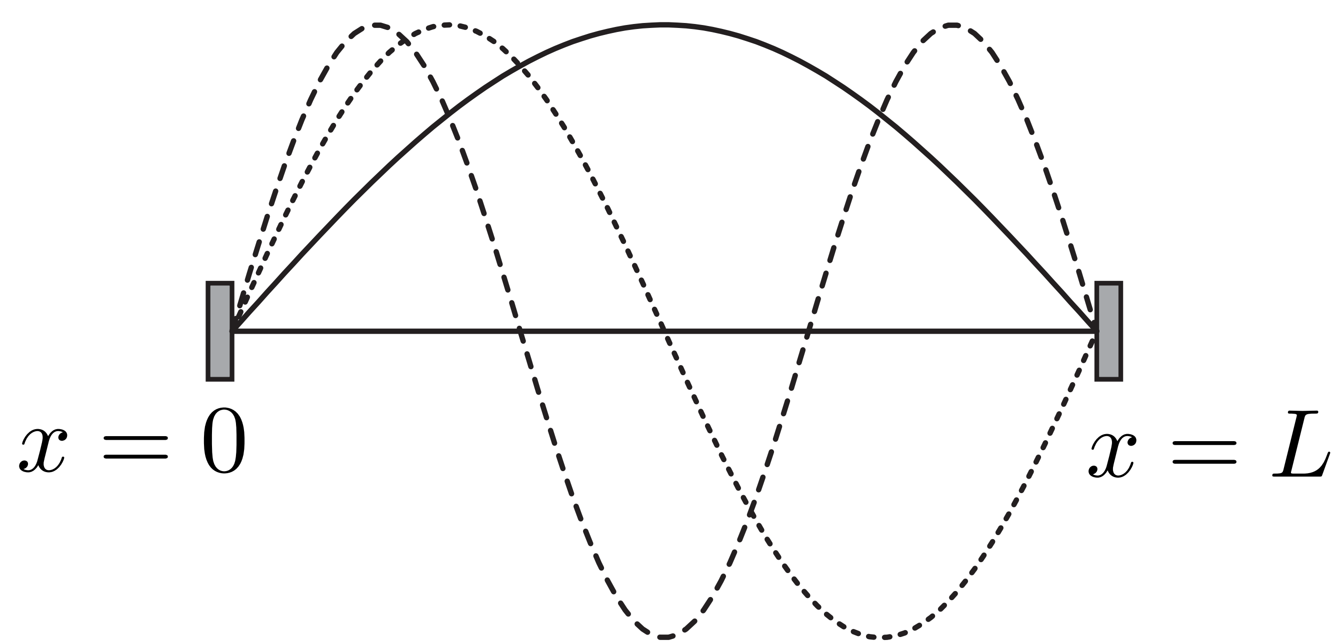
$$\rho \frac{\partial^2 u}{\partial t^2} = T \frac{\partial^2 u}{\partial x^2} \rightarrow -\rho\omega_n^2 u = -Tk_n^2 u \rightarrow \omega_n = (T/\rho)^{1/2} k_n = ck_n$$

Normal modes

- Solution:

$$u(x,t) = F(x-ct) - F(-x-ct) = \sum_{n=1}^{\infty} a_n \sin k_n x \cos \omega_n t$$

- Boundary conditions are automatically satisfied for all time
- Each mode is orthogonal to the others, and there is no energy transfer between them ($\dot{a}_n = 0$)



$$u(0,t) = u(L,t) = 0$$

Normal modes

- From the initial conditions $u(x,0) = \sum_{n=1}^{\infty} a_n \sin k_n x$

  determine components by overlap with each mode:

$$a_n = \frac{1}{L}\int_0^L dx\, u(x,0) \sin k_n x$$

- Complete behaviour at all subsequent times:

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{\pi n x}{L} \cos\frac{\pi n c t}{L}$$

Truncation errors

- Make tractable by putting a bound on the mode sums:

$$\sum_{n=1}^{\infty} \to \sum_{n=1}^{n_c}$$

- Spatial resolution $\Delta x = 2L/n_c$ is set by the wavelength $\lambda = 2\pi/k_n = 2L/n$ of the highest mode

- Discarded modes should have negligible power

$$E = \frac{1}{2}\int dx \left[\left(\frac{\partial u}{\partial t}\right)^2 + c^2\left(\frac{\partial u}{\partial x}\right)^2\right]$$

$$= \sum_{n=1}^{\infty} \omega_n |a_n|^2 \sim \sum_{n=1}^{n_c} n^2 |a_n|^2 + O((n_c+1)|a_{n_c+1}|)^2$$

Spectral method

- Alternative strategy for more general PDEs that support wavelike motion, e.g.,

$$\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} + f\frac{\partial^4 u}{\partial x^4} + \cdots$$

- We can start from the ansatz $u(x,t) = u_n(x)e^{i\omega_n t - \gamma t}$
- Solve the space-only conventional ODE for each mode:

$$(i\omega_n - \gamma)^2 u_n(x) + b(i\omega_n - \gamma)u_n(x) = c^2 u_n''(x) + f u_n''''(x) + \cdots$$

Spectral method

- View as a (real-valued, $\gamma = b/2$) eigenequation

$$\left(c^2 \frac{\partial^2}{\partial x^2} + f \frac{\partial^4}{\partial x^4}\right) u_n(x) = \left(-\omega_n^2 + \frac{1}{4} b^2\right) u_n(x)$$

- Imposition of boundary conditions leads to a discrete eigenspectrum, corresponding to the normal modes