Physics 750: Exercise 12

Thursday, October 26, 2017

1. Use the `curl` command to download from the class website everything you'll need for this exercise.

```
$ WEBPATH=http://www.phy.olemiss.edu/~kbeach/
$ curl $WEBPATH/courses/fall2017/phys750/src/exercise12.tgz -O
$ tar xzf exercise12.tgz
$ cd exercise12
```

2. The program `prod.cpp` computes the inner and outer products of several vectors using functions declared in the header file `linalg.h`. The corresponding function definitions (not yet written) are located in `linalg.cpp`. Fill in the bodies of the functions `inner_prod` and `outer_prod` so that they compute $(u, v) = \sum_i u_i v_i$ and $(u \otimes v)_{ij} = u_i v_j$, respectively. You should be able to reproduce the following session.

```
$ make prod
g++ -o prod prod.cpp -O2 -ansi -pedantic -Wall -lm
$ ./prod
u =
/   1 \
|  -2 |
| 0.5 |
\ 0.3 /

v =
/  -1 \
|   1 |
\   3 /

Inner product:
(u,u) = 5.34
(v,w) = 2

Outer product:
u.v =
/  -1   1   3 \
|   2  -2  -6 |
|-0.5 0.5 1.5 |
\-0.3 0.3 0.9 /
```

3. Recall that the determinant of a 2×2 matrix is given by

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

The determinant of an arbitrary $N \times N$ matrix can be expressed recursively in terms of the determinants of the $(N-1) \times (N-1)$ *minors*, which are the submatrices derived by removing one row and one column. For example, consider the row-1 version of Laplace's formula

$$\det M = \sum_{j=1}^{N} (-1)^{i+1} M_{1,j} \det M^{(1,j)},$$

where the minor $M^{(1,j)}$ holds the elements of M except those in the first row and the jth column. Let's show this explicitly for $N = 3$:

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a\det\begin{pmatrix} e & f \\ h & i \end{pmatrix} - b\det\begin{pmatrix} d & f \\ g & i \end{pmatrix} + c\det\begin{pmatrix} d & e \\ g & h \end{pmatrix}$$
$$= a(ei - fh) - b(di - fg) + c(dh - eg).$$

Implement this formula in the function `determinant3`

Matrix multiplication $C = AB$ can be evaluated element-wise according to $C_{ij} = \sum_k A_{ik}B_{kl}$, provided that the matrix dimensions of A and B are compatible. Have the function `multiply` carry out this computation.

Since the matrices are stored in row-major order, there is a special simplification when evaluating matrix multiplications of the form $C = AB^{\mathrm{t}}$, where t denotes the transpose. In that case, B^{t} is the *column-major* version of B, and the columns of B^{t} are accessible as the rows of B. Hence, the i, j element of $C = AB^{\mathrm{t}}$ can be viewed as an inner product of the ith row of A with the jth row of B: i.e., $C_{ij} = \sum_k A_{ik}B_{jk} = (a^{(i)}, b^{(j)})$. Write the body of `multiply_Rtranspose` so that it carries out this computation using only nested `for` loops and calls to `inner_prod`.

If everything is coded correctly, you should be able to reproduce the following session.

```
$ make det
g++ -o det det.cpp -O2 -ansi -pedantic -Wall -lm
$ ./det
A =
/   4  -3 \
\  -2   1 /
det(A) = -2

B =
/   3  -4   1 \
|   5   3  -7 |
\  -9   2   6 /
det(B) = 1

B*B =
/ -20 -22  37 \
|  93 -25 -58 |
\ -71  54  13 /
det(B*B) = 1

B*Bt =
/  26  -4 -29 \
|  -4  83 -81 |
\ -29 -81 121 /
det(B*Bt) = 1

C =
/   2  -5   6   1 \
|  -9  -7   1   4 |
\   8   2   9   3 /

C*Ct =
```

```
/  66  27  63 \
|  27 147 -65 |
\  63 -65 158 /
det(C*Ct) = 334311
```

4. Implement Laplace's determinant algorithm for square matrices of arbitrary order in `linalg.cpp` by completing the function `determinant`. Use the comments in the code to guide you. A good check is that your function should obey the relation $\det(AB) = \det(AB^{\mathrm{t}}) = \det(A)\det(B)$. The program `Laplace.cpp` will test this:

```
$ make
g++ -o Laplace Laplace.cpp -O2 -ansi -pedantic -Wall -lm
$ ./Laplace
A =
/   1   2   3   4   5   6 \
|   5   1   2   3   4   5 |
|  -1   0   1  -7   3   4 |
|  -2  12   0   1   2   3 |
|  -3  -3  -1   0  13   2 |
\  -4  -3  -5  -5   0   1 /
det(A) = 174375

B =
/   1   0   3   0   5   1 \
|   0   1   1   1   4   0 |
|  -1   0   1  -7   1   4 |
|  -2   1   0   1   0   3 |
|   0  -3  -1   0   1   2 |
\  -1  -1  -5  -1   0   0 /
det(B) = 625

AB =
/ -16 -15 -27 -21  21  35 \
|  -8 -13 -11 -15  35  30 |
|   8 -20 -25 -18  -1 -12 |
|  -7   4 -11  10  40   5 |
|  -4 -44 -36   2 -15  19 |
\  10  -9 -25  26 -37 -39 /
det(AB) = 108984375
det(A)*det(B) = 108984375

ABt =
/  41  29   3  22   8 -22 \
|  36  22   0   9   9 -19 |
|  21   6  70   7  10   3 |
|  11  21   9  26 -28 -11 |
|  61  48  23   9  27  11 |
\ -18 -13  38   3  16  37 /
det(ABt) = 108984375
det(A)*det(B) = 108984375
```