* Even if all fundamental laws and the complete zoology of particles are known, we can still only solve for a limited set of single- and few-body behaviours.
  → only a handful of 3-body problems are integrable!
  → related to the notion of computational irreducibility.

* What to do with $10^{27}$ interacting particles (a litre of water, say)?
  → Formulate coarse-grained theories in terms of macroscopic variables:
    e.g. average local values of particle or momentum densities, magnetization; fluctuations of such quantities or their response to external fields
  → observe and characterize the many different thermodynamically stable phases of matter:
    e.g. fluids flow
    solids are rigid
    Some matter is transparent ... others coloured
    transport of heat and change in insulators, metals and semiconductors.

* CUP provides a framework for understanding the properties of various phases of matter:

  1. Microscopic picture: large group of particles interacting via well-known (mostly Coulombic) forces.
(2) focus on macroscopic properties: rather than trajectories of individual particles, we use a local averaging
  → statistical mechanics + thermodynamics
  → macroscopic variables that vary slowly and continuously in space (classical + quantum continuum field theories)

(3) important connection to the organization of matter:
  geometric properties, patterns and regularity (or the lack of it)
  e.g. regular solid
      liquid
      glass

\[ | \cdot \cdot \cdot \rangle \quad | \cdot \cdot \cdot \rangle \quad | \cdot \cdot \cdot \rangle \]

↑ ordered arrangement

↓

many nearly equivalent low-energy configurations
⇒ almost no cost to deformation, hence a liquid has no shape

(4) unifying concepts: conservation laws (conserved quantities = constants of the motion)

and broken symmetries
e.g. in an isolated system, particle number, energy, and momentum are conserved.

- the system at sufficiently high temperature
  
  \[ \text{necessarily disordered, uncorrelated, homogeneous and isotropic} \]
  
  \[ \text{has full rotational and translational symmetry of free space} \]
  
  \[ \text{low-freq. long-wavelength behaviour controlled by hydrodynamical eqns} \]
  
  \[ \text{via cons. mass} \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]
  
  \[ \text{via cons. momentum} \quad \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho} \nabla P + \mathbf{F}_{\text{ext}} \]

- System at low temperature
  
  \[ \text{new thermodynamically stable phases condense, having progressively lower symmetry} \]

  \[ \text{e.g. a periodic crystal is unInvariant wrt a discrete set of spatial transformations} \]

  \[ \text{associated with each broken symmetry are distortions, defects, and dynamical modes (which provide a pathway to restoring the high-symmetry state)} \]
Order parameter theory of a crystal distortion

+ Suppose atoms in a solid feel a potential that stabilizes a cubic crystal structure; along a cleaved edge we see a square lattice with dimensions $a = b$

$\rightarrow$ Invariance under $90^\circ$ rotations means that the overall free energy must be invariant under the swap $(a, b) \rightarrow (b, a)$

$\rightarrow$ Hence $V(a, b) = V(b, a)$ or in an alternative set of coordinates $V(ab, b) = V(\phi)$ where $\phi = \frac{a}{b}$

and $V(\phi)$ is even in $\phi$ (since $\phi \rightarrow -\phi$ equals $(a, b) \rightarrow (b, a)$)

$\rightarrow$ For small distortions, expand in power series

$V(\phi) = A\phi^2 + B\phi^4 \quad (B > 0 \text{ for stability})$
\[ V(\phi) = 2A\phi + 4B\phi^3 \]
\[ = 2\phi(A + 2B\phi^2) = 0 \]

\[ \text{Stationary when } \phi = 0 \text{ or } \phi = \pm \sqrt{-\frac{A}{2B}} \]

Hence \((\phi, V)_{mm} = \begin{cases} (0, 0) & \text{for } A > 0 \\ \left( \pm \sqrt{\frac{-A}{2B}}, -\frac{A^2}{4B} \right) & \text{for } A < 0 \end{cases} \]

*Continuous symmetry breaking as \( A \) is tuned through zero*

\[ \phi_{mm} \]

\[ \text{distorted} \quad \text{undistorted} \]

\[ \rightarrow A, \text{ or some physical } \text{“dummy parameter”} \]

\[ \rightarrow A \text{ may have some dependence on physical quantities such as pressure or temperature} \]
local stability determined by a field \( \phi(\mathbf{r}) \), which serves as a position-dependent order parameter

\[
\phi(\mathbf{r}) = 0 \\
\phi(\mathbf{r}) > 0
\]

with an integrated free energy

\[
F[\phi] = \int d^2 \mathbf{r} \left( \frac{\xi}{2} \left( \nabla \phi \right)^2 + V(\phi) \right)
\]

functional of \( \phi(\mathbf{r}) \) encodes the elastic energy cost of mismatches between horizontally and vertically deformed domains; \( \xi \) is the stiffness of deformation.

width of domain wall / grain boundary
Diffusive dynamics (roughly) given by
\[
\frac{\partial \phi}{\partial t} = -\frac{\partial F[\phi]}{\partial \phi} = \phi_s \nabla^2 \phi + 2A\phi - 4B\phi^3
\]

→ simple model of competition between two competing phases

More possibilities when the order parameter is not a scalar
(e.g. complex number or a vector)

\[V(\phi) = A\phi + B\phi^4\]

Two ground states connected by barrier tunnelling

\[V(\phi) = A|\phi|^2 + B|\phi|^4\]

Real \(\phi\)

\(U(1)\) degenerate family of ground states