Physics 451: Introduction to Quantum Mechanics

First In-class Test

Thursday, October 8, 2015 / 09:30–10:45 / Room 228, Lewis Hall

Student's Name: ______________________

Instructions

There are nine questions. You should attempt all of them. Mark your response on the test paper in the space provided. Please use a pen. If in answering a question you sketch a diagram, please provide meaningful labels. Aids of any kind—including class notes, textbooks, cheat sheets, and calculators—are not permitted.

Good luck!

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20 points
Fourier Transforms

\[ \psi(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tilde{\psi}(k) \]

\[ \tilde{\psi}(k) = \int_{-\infty}^{\infty} dx \ e^{-ikx} \psi(x) \]

\[ \frac{\sin qx}{\pi x} = \int_{-q}^{q} \frac{dk}{2\pi} e^{ikx} \]

\[ \frac{e^{-x^2/2\alpha}}{\sqrt{2\pi\alpha}} = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} e^{-\alpha k^2/2} \]

Trigonometric identities

\[ 2 \cos \theta = e^{i\theta} + e^{-i\theta} \]

\[ 2i \sin \theta = e^{i\theta} - e^{-i\theta} \]
Short answer questions (4 points)

1. The following has the form of a classical wave equation. Make the minimal number of modifications required to turn it into the \textit{time-dependent} Schrödinger equation.

\[
\frac{\hbar}{\omega} \frac{\partial^2 \psi(x,t)}{\partial t^2} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\psi(x,t)
\]

2. We studied the case of a single particle in one dimension, constrained by a square well potential of finite width \(L\) and finite depth \(V_0\) (which we can take to be centered on the origin). Which of the following is an incorrect statement.

(a) The eigenstates of the system are states of definite parity.

(b) The bound states have quantized energy levels given by the solution of a transcendental equation.

(c) There are always at least two bound states, regardless of the values of \(L\) and \(V_0\).

(d) In the limit where \(L \to 0\) and \(V_0 \to \infty\) with \(LV_0\) held fixed, the potential approaches a delta function; there is a single bound eigenstate \(\phi(x)\), and \(\phi'(x)\) is discontinuous at \(x = 0\).

3. The eigenstates of the infinite square well \(\{|n\rangle : n = 1, 2, 3, \ldots\}\) have a real-space representation

\[
\phi_n(x) = \langle x|n\rangle = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right).
\]

Any arbitrary wave function inside the well can be expanded as

\[
\psi(x) = \sum_{n=1}^{\infty} \langle n|\psi\rangle \phi_n(x).
\]

This is a statement of what?

(a) orthogonality

(b) orthonormality

(c) completeness

(d) duality

4. The wave function \(\psi(r) \sim \exp(ik \cdot r)\) describes a particle that is confined to an \(L \times L \times L\) box but otherwise free; i.e., \(V(r) = 0\) inside the box and \(V(r) = \infty\) outside. Which of the following is the correct normalization factor?

(a) \(L^{-3/2}\)

(b) \(L^{-2/3}\)

(c) \(1/L\)

(d) \(1/\sqrt{L}\)

(e) \(L^{1/2}\)
Long answer questions (21 points)

Questions 5–7: Consider the (properly normalized) wave function

$$\psi(x) = e^{ik_0x} \frac{1}{\sqrt{\pi k_1}} \frac{\sin k_1 x}{x}.$$

5. Compute the probability density for finding the particle at position $x$. Provide a sketch with properly labelled axes.

6. Compute the probability current

$$j(x) = \frac{\hbar}{2mi} \left( \psi \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x} \psi \right) = \frac{\hbar}{m} \cdot \text{Imag} \left( \psi \frac{\partial \psi}{\partial x} \right),$$

and show that the result is proportional to the velocity $\hbar k_0/m$. 
7. The Fourier Transform of $\psi(x)$ is related to the wave function in the momentum representation

$$\tilde{\psi}(k) \sim \begin{cases} 
1 & \text{if } k \in (-|k_1| + k_0, |k_1| + k_0) \\
0 & \text{otherwise}
\end{cases}$$

Write down and then evaluate the integral expressions for the expectation values $\langle \hat{p} \rangle$ and $\langle \hat{p}^2 \rangle$. You should find a mean momentum $\langle \hat{p} \rangle = \hbar k_0$ and variance $\sigma_p = (\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2)^{1/2} = \hbar |k_1|/\sqrt{3}$. 
Questions 8–9: A quantum system lives in one of three states, $|a\rangle$, $|b\rangle$, or $|c\rangle$. You can assume they form an orthonormal set. The system is prepared (at time $t = 0$) in a superposition

$$|\psi\rangle = |\psi(0)\rangle = \frac{1}{\sqrt{3}}|a\rangle + \frac{e^{i\beta}}{\sqrt{3}}|b\rangle + \frac{e^{i\gamma}}{\sqrt{3}}|c\rangle.$$ 

and evolves according to a hamiltonian

$$\hat{H} = \hbar \omega_a |a\rangle\langle a| - \hbar \omega_{bc} (|b\rangle\langle c| + |c\rangle\langle b|).$$

8. What are the probabilities of measuring each of $a$, $b$, and $c$ at time $t = 0$?

9. Solve for the time-evolved state $|\psi(t)\rangle = \exp(-i \hat{H}t/\hbar)|\psi(0)\rangle$. You should find that

$$|\psi(t)\rangle = \frac{e^{-i\omega_a t}}{\sqrt{3}}|a\rangle + \frac{e^{i\beta} \cos \omega_{bc} t + ie^{i\gamma} \sin \omega_{bc} t}{\sqrt{3}}|b\rangle + \frac{ie^{i\beta} \sin \omega_{bc} t + e^{i\gamma} \cos \omega_{bc} t}{\sqrt{3}}|c\rangle.$$