# Physics 451: Introduction to Quantum Mechanics

First In-class Test

Thursday, October 8, 2015 / 09:30–10:45 / Room 228, Lewis Hall

Student's Name: _____

## Instructions

There are nine questions. You should attempt all of them. Mark your response on the test paper in the space provided. **Please use a pen**. If in answering a question you sketch a diagram, please provide meaningful labels. Aids of any kind—including class notes, textbooks, cheat sheets, and calculators—are not permitted.

Good luck!

| | | | | |
|---|---|---|---|---|
| 4 | points | short answer | questions | 1–4 |
| 2 | | long answer | | 5 |
| 3 | | | | 6 |
| 4 | | | | 7 |
| 2 | | | | 8 |
| 5 | | | | 9 |
| **20** | **points** | | | |

## Fourier Transforms

$$
\begin{aligned}
\psi(x) &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \tilde{\psi}(k) \\
\tilde{\psi}(k) &= \int_{-\infty}^{\infty} dx\, e^{-ikx} \psi(x) \\
\frac{\sin qx}{\pi x} &= \int_{-q}^{q} \frac{dk}{2\pi}\, e^{ikx} \\
\frac{e^{-x^2/2\alpha}}{\sqrt{2\pi\alpha}} &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} e^{-\alpha k^2/2}
\end{aligned}
$$

## Trigonometric identities

$$
\begin{aligned}
2\cos\theta &= e^{i\theta} + e^{-i\theta} \\
2i\sin\theta &= e^{i\theta} - e^{-i\theta}
\end{aligned}
$$

## Short answer questions (4 points)

1. The following has the form of a classical wave equation. Make the minimal number of modifications required to turn it into the *time-dependent* Schrödinger equation.

$$\frac{\hbar}{\omega}\frac{\partial^2\psi(x,t)}{\partial t^2} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x)\psi(x,t)$$

2. We studied the case of a single particle in one dimension, constrained by a square well potential of finite width $L$ and finite depth $V_0$ (which we can take to be centered on the origin). Which of the following is an <u>incorrect</u> statement.

   (a) The eigenstates of the system are states of definite parity.

   (b) The bound states have quantized energy levels given by the solution of a transcendental equation.

   (c) There are always at least two bound states, regardless of the values of $L$ and $V_0$.

   (d) In the limit where $L \to 0$ and $V_0 \to \infty$ with $LV_0$ held fixed, the potential approaches a delta function; there is a single bound eigenstate $\phi(x)$, and $\phi'(x)$ is discontinuous at $x = 0$.

3. The eigenstates of the infinite square well $\{|n\rangle : n = 1, 2, 3, \ldots\}$ have a real-space representation

$$\phi_n(x) = \langle x|n\rangle = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right).$$

   Any arbitrary wave function inside the well can be expanded as

$$\psi(x) = \sum_{n=1}^{\infty}\langle n|\psi\rangle\phi_n(x).$$

   This is a statement of what?

   (a) orthogonality

   (b) orthonormality

   (c) completeness

   (d) duality

4. The wave function $\psi(\mathbf{r}) \sim \exp(i\mathbf{k}\cdot\mathbf{r})$ describes a particle that is confined to an $L \times L \times L$ box but otherwise free; i.e., $V(\mathbf{r}) = 0$ inside the box and $V(\mathbf{r}) = \infty$ outside. Which of the following is the <u>correct</u> normalization factor?

   (a) $L^{-3/2}$

   (b) $L^{-2/3}$

   (c) $1/L$

   (d) $1/\sqrt{L}$

   (e) $L^{1/2}$

## Long answer questions (21 points)

Questions 5–7: Consider the (properly normalized) wave function

$$\psi(x) = e^{ik_0 x} \frac{1}{\sqrt{\pi |k_1|}} \frac{\sin k_1 x}{x}.$$

5. Compute the probability density for finding the particle at position $x$. Provide a sketch with properly labelled axes.

6. Compute the probability current

$$j(x) = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \frac{\partial \psi^*}{\partial x}\psi\right) = \frac{\hbar}{m} \cdot \mathrm{Imag}\left(\psi^* \frac{\partial \psi}{\partial x}\right),$$

and show that the result is proportional to the velocity $\hbar k_0/m$.

7. The Fourier Transform of $\psi(x)$ is related to the wave function in the momentum representation

$$\tilde{\psi}(k) \sim \begin{cases} 1 & \text{if } k \in (-|k_1| + k_0, |k_1| + k_0) \\ 0 & \text{otherwise} \end{cases}$$

Write down and then evaluate the integral expressions for the expectation values $\langle \hat{p} \rangle$ and $\langle \hat{p}^2 \rangle$. You should find a mean momentum $\langle \hat{p} \rangle = \hbar k_0$ and variance $\sigma_p = (\langle \hat{p}^2 \rangle - \langle \hat{p} \rangle^2)^{1/2} = \hbar |k_1| / \sqrt{3}$.

Questions 8–9: A quantum system lives in one of three states, $|a\rangle$, $|b\rangle$, or $|c\rangle$. You can assume they form an orthonormal set. The system is prepared (at time $t = 0$) in a superposition

$$|\psi\rangle = |\psi(0)\rangle = \frac{1}{\sqrt{3}}|a\rangle + \frac{e^{i\beta}}{\sqrt{3}}|b\rangle + \frac{e^{i\gamma}}{\sqrt{3}}|c\rangle.$$

and evolves according to a hamiltonian

$$\hat{H} = \hbar\omega_a|a\rangle\langle a| - \hbar\omega_{bc}(|b\rangle\langle c| + |c\rangle\langle b|).$$

8. What are the probabilities of measuring each of $a$, $b$, and $c$ at time $t = 0$?

9. Solve for the time-evolved state $|\psi(t)\rangle = \exp(-i\hat{H}t/\hbar)|\psi(0)\rangle$. You should find that

$$|\psi(t)\rangle = \frac{e^{-i\omega_a t}}{\sqrt{3}}|a\rangle + \frac{e^{i\beta}\cos\omega_{bc}t + ie^{i\gamma}\sin\omega_{bc}t}{\sqrt{3}}|b\rangle + \frac{ie^{i\beta}\sin\omega_{bc}t + e^{i\gamma}\cos\omega_{bc}t}{\sqrt{3}}|c\rangle.$$

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