Physics 711: Assignment 4

(to be submitted by Friday, December 2, 2016)

In class, we considered a Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V} = \sum_{n=0}^{\infty} \epsilon_n |n\rangle\langle n| + \sum_{n=0}^{\infty} \Big(\Delta_n |n\rangle\langle n+1| + \Delta_n^* |n+1\rangle\langle n| \Big),$$

which describes a hierarchy of energy levels, each coupled to the levels immediately above and below. Let's assume that $\epsilon_n = \epsilon_0 \lfloor (n+3)/2 \rfloor$ and $\Delta_n = (\epsilon_0/2)(i/2)^n$. Then, so long as the states $\{|n\rangle\}$ form an orthonormal basis, the Hamiltonian has a matrix description

$$H = \begin{pmatrix} \epsilon_0 & \Delta_0 & 0 & 0 & 0 & \cdots \\ \Delta_0^* & \epsilon_1 & \Delta_1 & 0 & 0 & \\ 0 & \Delta_1^* & \epsilon_2 & \Delta_2 & 0 & \\ 0 & 0 & \Delta_2^* & \epsilon_3 & \Delta_3 & \\ 0 & 0 & 0 & \Delta_3^* & \epsilon_4 & \\ \vdots & & & & & \ddots \end{pmatrix} = \epsilon_0 \begin{pmatrix} 1 & 1/2 & & & & \\ 1/2 & 2 & i/4 & & & \\ & -i/4 & 2 & -1/8 & & \\ & & -1/8 & 3 & -i/16 & \\ & & & +i/16 & 3 & \\ & & & & & \ddots \end{pmatrix}$$

1. Break the energy degeneracies by diagonalizing within each degenerate 2×2 block.
2. Parameterize the Hamiltonian as $\hat{H} = \hat{H}_{\text{diag}} + \lambda \hat{H}_{\text{offdiag}}$. Use nondegenerate perturbation theory to compute the energy shift in the ground state to order λ^2 around the $\lambda = 0$ solution.

A one-dimensional potential takes the value $V(x) = (K/2)x^2$ on the interval $-L/2 < x < L/2$ but infinity everywhere else.

3. Sketch out how you would solve for the system's energy eigenstates using the shooting method.
4. Explain why

$$\psi_n(x; a_n) \sim \begin{cases} \exp[-(r/a_n)^2] \cos n\pi x/L & \text{if } n \text{ is even} \\ \exp[-(r/a_n)^2] \sin n\pi x/L & \text{if } n \text{ is odd} \end{cases}$$

   is a reasonable choice of trial wave function.

5. For the first few quantum numbers $n = 1, 2, 3, \ldots$, solve for the optimal value a_n that minimizes the energy estimate

$$E_n = \min_{a_n} \frac{\langle \psi_n(a_n) | \hat{H} | \psi_n(a_n) \rangle}{\langle \psi_n(a_n) | \psi_n(a_n) \rangle}.$$