Physics 711: Assignment 3
(to be submitted by Monday, November 14, 2016)

The Hamiltonian of the three-dimension quantum harmonic oscillator—describing a particle of mass $m$ in an isotropic Hooke’s law potential—can be decomposed into three one-dimensional harmonic oscillators:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2} = \hbar \omega \left( a_x^+ a_x + a_y^+ a_y + a_z^+ a_z + \frac{3}{2} \right) \equiv \hbar \omega \left( \hat{N} + \frac{3}{2} \right).$$

Here, $a_x$, $a_y$, and $a_z$ are the annihilation operators for a quantum of vibrational motion in the three orthogonal directions. It is convenient to define a basis of Cartesian states that satisfy an isotropic Hooke’s law potential—can be decomposed into three one-dimensional harmonic oscillators:

$$|n_x, n_y, n_z\rangle = \frac{1}{\sqrt{n_x!n_y!n_z!}} (a_x^+)^{n_x} (a_y^+)^{n_y} (a_z^+)^{n_z} |0, 0, 0\rangle$$

that satisfy $\hat{H}|n_x, n_y, n_z\rangle = \hbar \omega (n_x + n_y + n_z + 3/2)|n_x, n_y, n_z\rangle$.

1. List all the Cartesian states that have energy eigenvalues below $4\hbar \omega$. Indicate the degeneracies.

2. Find a general expression for the degeneracy of the states with total occupation $N$.

3. Show that the orbital angular momentum operator $\hat{L} = (\hat{L}_x, \hat{L}_y, \hat{L}_z)$ and its magnitude can be expressed as follows:

$$\hat{L}_i = i\hbar \sum_{jk} \epsilon_{ijk} a_j^+ a_k, \quad \hat{L}^2 = \hbar^2 \left[ \hat{N} (\hat{N} + 1) - \sum_{jk} a_j^+ a_j a_k a_k \right].$$

4. Find states $|n, l, m\rangle$ such that $\hat{L}^2 |n, l, m\rangle = \hbar l (l + 1)$ and $\hat{L}_z |n, l, m\rangle = \hbar m$ with $m = -l, -l + 1, \ldots, l - 1, l$. Work this out for $l = 0, 1, 2$. Explain why the $l \geq 2$ cases are different from $l = 0$ and $l = 1$.

The final four questions are concerned with the connection between SU(2) and SO(3). Recall that for every $U \in SU(2)$, there is a corresponding $3 \times 3$ rotation matrix $R^{ab} = \frac{1}{2} \text{Tr} \sigma^a U \sigma^b U^\dagger$.

5. For a unit vector $\mathbf{n}$ in three-dimensional space, demonstrate that

$$U(\phi \mathbf{n}) = e^{-i(\phi/2)\mathbf{n} \cdot \sigma} = I \cos \frac{\phi}{2} - i \mathbf{n} \cdot \sigma \sin \frac{\phi}{2}$$

and that $U(\phi \mathbf{n}) \in SU(2)$.

6. Prove the following four identities involving traces of the Pauli matrices: $\text{Tr} \sigma^a = 0$, $\text{Tr} \sigma^a \sigma^b = 2 \delta^{ab}$, $\text{Tr} \sigma^a \sigma^b \sigma^c = 2i \varepsilon^{abc}$, $\text{Tr} \sigma^a \sigma^b \sigma^c \sigma^d = 2 (\delta^{ab} \delta^{cd} - \delta^{ac} \delta^{bd} + \delta^{ad} \delta^{bc})$.

7. Show that the $3 \times 3$ rotation matrix arising from $U(\phi \mathbf{n})$ has the form

$$R^{ab} = \frac{1}{2} \text{Tr} \sigma^a U \sigma^b U^\dagger = \delta^{ab} \cos \phi + (1 - \cos \phi) n^a n^b - \varepsilon^{abc} n^c \sin \phi.$$

Compute $R(\phi \mathbf{e}_x)$ and $R(\phi \mathbf{e}_z)$.

8. Show that the action of two rotations $R_1 = R(\phi \mathbf{n})$ and $R_2 = R(\theta \mathbf{m})$, applied in succession, is characterized by product of the corresponding unitary matrices $U_2 = U(\theta \mathbf{m}) = e^{i(\theta/2)\mathbf{m} \cdot \sigma}$ and $U_1 = U(\phi \mathbf{n}) = e^{i(\phi/2)\mathbf{n} \cdot \sigma}$:

$$\sum_b R^{ab}_2 R^{bc}_1 = \frac{1}{2} \text{Tr} \sigma^a U_2 U_1 \sigma^c U_1^\dagger U_2^\dagger.$$