Physics 451: Introduction to Quantum Mechanics

Second In-class Test

Tuesday, November 4, 2014 / 09:30–10:45 / Room 2-228, Lewis Hall

Student's Name: ___________________

Instructions

There are 14 questions. You should attempt all of them. Mark your response on the test paper in the space provided. Aids of any kind—including class notes, textbooks, cheat sheets, and calculators—are not permitted.

Good luck!

7	points	short answer	questions	1–7
1		long answer		8
4				9
2				10
3				11
2				12
3				13
3				14
25	**points**			

Trigonometric identities

$$2\cos\theta = e^{i\theta} + e^{-i\theta}$$
$$2i\sin\theta = e^{i\theta} - e^{-i\theta}$$

Linear algebra identities

$$\begin{aligned} \text{a matrix} \qquad & M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ \text{its inverse} \qquad & M^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\ \text{its characteristic polynomial} \qquad & p_M(\lambda) = \det(M - \lambda I) = (a - \lambda)(d - \lambda) - bc \end{aligned}$$

Algebra of the ladder operators (quantum harmonic oscillator)

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$
$$a|n\rangle = \sqrt{n}|n-1\rangle$$

Short answer questions (7 points)

1. A quantum harmonic oscillator of mass m and natural frequency ω has eigenstates $\{|\phi_n\rangle\}$. The overlap between any two such states satisfies the following orthogonality relation:

$$\langle\phi_n|\phi_{n'}\rangle = \sqrt{\frac{m\omega}{2^{n+n'}n!(n')!\pi\hbar}} \int_{-\infty}^{\infty} dx\, e^{-m\omega x^2/\hbar} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) H_{n'}\left(\sqrt{\frac{m\omega}{\hbar}}x\right) = \begin{cases} 1 & \text{if } n = n', \\ 0 & \text{otherwise.} \end{cases}$$

What does the symbol H_n represent here?

(**A**) Hermite polynomial

(b) Hildebrandt polynomial

(c) Hilbert space

(d) trace of the Hamiltonian, $\mathrm{Tr}\, \hat{H}$

(e) expectation value of the Hamiltonian in the nth state, $\langle\phi_n|\hat{H}|\phi_n\rangle$

2. An electron feels an image charge potential that attracts it to a metallic surface. (The potential is zero when the electron is an infinite distance from the surface.) In class, we computed the energy levels for this system: $E_n = -(0.85\,\mathrm{eV})/n^2$ (with $n = 1, 2, 3, \ldots$). The ground state ($n = 1$) wave function is $\phi_1(z < 0) = 0$ and $\phi_1(z > 0) \sim (z/a_B)e^{-z/a_B}$. Which of the following is an incorrect statement?

(a) In each mode n, the wave function $\phi_n(z)$ vanishes at the metal surface ($z = 0$).

(b) The wave function $\phi_n(z)$ has $n - 1$ nodes at positions $z > 0$ away from the surface.

(c) The expectation value $\langle\phi_1|z|\phi_1\rangle$ is greater than zero and has units of a_B.

(d) There is a countably infinite number of bound states.

(e) There is a continuum of postive-energy states with wave functions that behave like e^{ikz} at distances far from the surface.

(**F**) There is a finite number of states with energy $E < 0$.

3. Consider the potential $V(x < 0) = 0$ and $V(x > 0) = V_0$, representing a step-edge barrier of height V_0. The wave function $\psi(x)$ describes the situation in which a flux of particles with energy E is incident from the negative-x side. Which of the following is an incorrect statement?

(a) In the case where $0 < E < V_0$, the probability $\int_0^\infty dx\, |\psi(x)|^2$ is greater than zero, even though there is *no* probability current in the region $x > 0$.

(b) In the case where $E > V_0$, the reflection probability is $R = [(k_+ - k_-)/(k_+ + k_-)]^2$, with k_- and k_+ denoting the wave vectors in the regions $x < 0$ and $x > 0$.

(**C**) In the case where $E > V_0$, the transmission probability is $T = (k_+^2 - k_-^2)/(k_+ + k_-)^2$.

4. At time $t = 0$, an electron is described by a gaussian wave packet $\phi(x,0) \sim \exp[-\frac{1}{2}(x/\sigma)^2]$. Which of the following is an incorrect statement.

   (a) The expectation value $\langle\phi(t)|\hat{x}|\phi(t)\rangle$ is zero for all time.

   (b) The corresponding k-space description of the wave packet is produced by Fourier transformation.

   (**C**) The wave function $\phi(x,0)$ of width $\sim \sigma$ has a transform pair $\tilde{\phi}(k,0)$ of width $\sim e^{-\sigma^2}$.

   (d) Because of the nonlinear dispersion relation (frequency ω as a function of k), the wave packet distorts as it propagates in time.

   (e) Measuring time in appropriately chosen units, we find that the effective width of the wave packet evolves according to $\sigma(t) = \sigma(0)\sqrt{1+t^2}$.

5. Which of the following is an incorrect statement about the wave function $\psi(\mathbf{r}) \sim \exp(i\mathbf{k}\cdot\mathbf{r})$.

   (a) $\psi(\mathbf{r})$ is a plane wave state.

   (b) $\psi(\mathbf{r})$ represents a state of definite momentum $\mathbf{p} = \hbar\mathbf{k}$.

   (**C**) The wave function of an energy eigenstate can behave like $\psi(\mathbf{r}) \sim \exp(i\mathbf{k}\cdot\mathbf{r})$ only in regions where the potential is exactly zero.

   (d) The probability density $|\psi(\mathbf{r})|^2$ of finding the particle in the vicinity of $\mathbf{r}$ is uniform in space.

6. Which of the following has the same asymptotic behavior (both $x \to -\infty$ and $x \to \infty$) as the wave function for a bound state in a finite square well? Suppose that the well is defined by $V(x) = 0$ for $|x| < L/2$ and $V(x) = V_0$ for $|x| > L/2$.

   (a) $L^2/(L^2 + x^2)$

   (b) $\exp[-(2mE/\hbar^2)x^2]$

   (**C**) $\exp(-[2m(V_0 - E)/\hbar^2]^{1/2}|x|)$

   (d) $\exp[-(2mE/\hbar^2)^{1/2}x]$

   (e) $1/\cosh[(2mE/\hbar^2)^{1/2}x]$

7. When written in terms of the creation and annihilation operators, $a^\dagger$ and a, the quantum harmonic oscillator Hamiltonian has this compact form: $\hat{H} = \hbar\omega(a^\dagger a + 1/2)$. The energy eigenstates $\{|n\rangle : n = 1,2,3,\ldots\}$ are eigenstates of the number operator $\hat{N} = a^\dagger a$. Which two of the following expressions are incorrect. (Circle both.)

   (a) $a^\dagger a a^\dagger a|n\rangle = n^2|n\rangle$

   (**B**) $aa^\dagger + a^\dagger a = 1$

   (c) $|n\rangle = (1/\sqrt{n!})(a^\dagger)^n|0\rangle$

   (d) $a^\dagger|0\rangle = |1\rangle$

   (e) $a|1\rangle = |0\rangle$

   (**F**) $a|0\rangle = |-1\rangle$

Long answer questions (21 points)

Consider the Hamiltonian

$$\hat{H} = \epsilon + Va^\dagger + V^* a + Ua^\dagger a(a^\dagger a - 1).$$

As usual, $a^\dagger$ and a are the quantum harmonic oscillator creation and annihilation operators.

8. Explain why both ϵ and U must be real.

   To ensure real energy eigenvalues, the Hamiltonian must be hermitian ($\hat{H}^\dagger = \hat{H}$), which is the same as demanding that $\epsilon = \epsilon^*$ and $U = U^*$.

9. Provide the eight matrix elements $H_{mn} = \langle m|\hat{H}|n\rangle$ that are missing (marked "?") from the matrix below.

$$H = \begin{pmatrix} H_{00} & H_{01} & H_{02} & \cdots \\ H_{10} & H_{11} & & \\ H_{20} & & \ddots & \\ \vdots & & & \end{pmatrix} = \begin{pmatrix} \epsilon & V^* & 0 & 0 & \cdots \\ V & \epsilon & \sqrt{2}V^* & 0 & \\ 0 & \sqrt{2}V & \epsilon + U & \sqrt{3}V^* & \\ 0 & 0 & \sqrt{3}V & \epsilon + 6U & \\ \vdots & & & & \ddots \end{pmatrix}.$$

In the limit where U is the largest energy scale in the system by far, the low-energy eigenstates live within the subspace spanned by just $|0\rangle$ and $|1\rangle = a^\dagger|0\rangle$

$$H \simeq \begin{pmatrix} H_{00} & H_{01} \\ H_{10} & H_{11} \end{pmatrix} = \begin{pmatrix} \epsilon & |V|e^{-i\alpha} \\ |V|e^{i\alpha} & \epsilon \end{pmatrix}.$$

Note that we've written $V = |V|e^{i\alpha}$ with the complex phase made explicit.

10. Properly normalize the states $|\psi_+\rangle \sim |0\rangle + e^{i\alpha}|1\rangle$ and $|\psi_-\rangle \sim |0\rangle - e^{i\alpha}|1\rangle$.

    Define $|\psi_\pm\rangle = C_\pm(|0\rangle \pm e^{i\alpha}|1\rangle)$ and demand that

$$\begin{aligned} \langle\psi_\pm|\psi_\pm\rangle &= |C_\pm|^2 (\langle 0| \pm e^{-i\alpha}\langle 1|)(|0\rangle \pm e^{i\alpha}|1\rangle) \\ &= |C_\pm|^2 (\langle 0|0\rangle + e^{i(\alpha-\alpha)}\langle 1|1\rangle) \\ &= 2|C_\pm|^2 = 1. \end{aligned}$$

    Hence, $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm e^{i\alpha}|1\rangle)$.

11. Show that $|\psi_+\rangle$ and $|\psi_-\rangle$ are eigenstates of the Hamiltonian. Determine the corresponding energy eigenvalues, expressed in terms of ϵ and $|V|$.

$$\begin{pmatrix} \epsilon & |V|e^{-i\alpha} \\ |V|e^{i\alpha} & \epsilon \end{pmatrix}\begin{pmatrix} 1 \\ e^{i\alpha} \end{pmatrix} = \begin{pmatrix} 1 \\ \pm e^{i\alpha} \end{pmatrix} = \begin{pmatrix} \epsilon + |V| \\ |V|e^{i\alpha} \pm \epsilon e^{i\alpha} \end{pmatrix} = (\epsilon \pm |V|)\begin{pmatrix} 1 \\ \pm e^{i\alpha} \end{pmatrix}$$

12. Prove that

$$|0\rangle = \frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle),$$
$$|1\rangle = \frac{e^{-i\alpha}}{\sqrt{2}}(|\psi_+\rangle - |\psi_-\rangle).$$

Add

$$|\psi_+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{i\alpha}|1\rangle),$$
$$|\psi_-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - e^{i\alpha}|1\rangle)$$

to get $|\psi_+\rangle + |\psi_-\rangle = (2/\sqrt{2})|0\rangle = \sqrt{2}|0\rangle$. Subtract

$$|e^{-i\alpha}\psi_+\rangle = \frac{1}{\sqrt{2}}(e^{-i\alpha}|0\rangle + |1\rangle),$$
$$|e^{-i\alpha}\psi_-\rangle = \frac{1}{\sqrt{2}}(e^{-i\alpha}|0\rangle - |1\rangle)$$

to get $e^{-i\alpha}(|\psi_+\rangle - |\psi_-\rangle) = \sqrt{2}|1\rangle$.

13. The system is prepared in an initial state $|\psi(0)\rangle = \frac{2}{\sqrt{5}}|0\rangle + \frac{i}{\sqrt{5}}|1\rangle$. Compute the time dependence of the state according to $|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|\psi(0)\rangle$. You should find that

$$|\psi(t)\rangle = \frac{e^{-i\epsilon t/\hbar}}{\sqrt{10}} \sum_{\eta=\pm 1} (2 + i\eta e^{-i\alpha}) e^{-i\eta|V|t/\hbar}|\psi_\eta\rangle.$$

$$\begin{aligned}
|\psi(t)\rangle &= e^{-i\hat{H}t/\hbar}\left(\frac{2}{\sqrt{5}}|0\rangle + \frac{i}{\sqrt{5}}|1\rangle\right) \\
&= e^{-i\hat{H}t/\hbar}\left(\frac{2}{\sqrt{5}}\left[\frac{1}{\sqrt{2}}(|\psi_+\rangle + |\psi_-\rangle)\right] + \frac{i}{\sqrt{5}}\left[\frac{e^{-i\alpha}}{\sqrt{2}}(|\psi_+\rangle - |\psi_-\rangle)\right]\right) \\
&= e^{-i\hat{H}t/\hbar}\left(\frac{2 + ie^{-i\alpha}}{\sqrt{10}}|\psi_+\rangle + \frac{2 - ie^{-i\alpha}}{\sqrt{10}}|\psi_-\rangle\right) \\
&= \frac{1}{\sqrt{10}}\left((2 + ie^{-i\alpha})e^{-i(\epsilon+|V|)t/\hbar}|\psi_+\rangle + (2 - ie^{-i\alpha})e^{-i(\epsilon-|V|)t/\hbar}|\psi_-\rangle\right) \\
&= \frac{e^{-i\epsilon t/\hbar}}{\sqrt{10}}\left((2 + ie^{-i\alpha})e^{-i|V|t/\hbar}|\psi_+\rangle + (2 - ie^{-i\alpha})e^{+i|V|t/\hbar}|\psi_-\rangle\right) \\
&= \frac{e^{-i\epsilon t/\hbar}}{\sqrt{10}} \sum_{\eta=\pm 1} (2 + i\eta e^{-i\alpha})e^{-i\eta|V|t/\hbar}|\psi_\eta\rangle
\end{aligned}$$

14. Show explicitly that $|\psi(t)\rangle$ retains its unit normalization for all time.

Since $|\psi_+\rangle$ and $|\psi_-\rangle$ are orthogonal to one another,

$$\begin{aligned}
\langle\psi(t)|\psi(t)\rangle &= \frac{1}{10}\sum_{\eta=\pm1}(2-i\eta e^{+i\alpha})e^{+i\eta|V|t/\hbar}\langle\psi_\eta|\cdot\sum_{\eta'=\pm1}(2+i\eta' e^{-i\alpha})e^{-i\eta'|V|t/\hbar}|\psi'_\eta\rangle \\
&= \frac{1}{10}\sum_{\eta=\pm1}\sum_{\eta'=\pm1}(2-i\eta e^{+i\alpha})\cdot(2+i\eta' e^{-i\alpha})e^{i(\eta-\eta')|V|t/\hbar}\underbrace{\langle\psi_\eta|\psi'_\eta\rangle}_{=\delta_{\eta,\eta'}} \\
&= \frac{1}{10}\sum_{\eta=\pm1}(4-i^2\eta^2) \\
&= \frac{2\cdot5}{10} = 1
\end{aligned}$$