

Name:

Section:

Date:

Physics 221

Lab Physics for Science and Engineering I

Measurement and Uncertainty PART 2

THEORY

Purpose

Students will learn a general method for propagating error using differential calculus. This method will be employed in determining the density of an object.

Introduction

Last week, the “half min-max” method was used to determine the uncertainty in the volume and density of an object. While the idea behind this method is fairly straightforward to understand, it provides only a maximum estimate of the uncertainty, which is not ideal. Although an overestimation of the uncertainty of a measurement is preferable to an underestimation, a more accurate method which minimizes uncertainty (while still not underestimating) should be employed.

Using differential calculus, we will be able to propagate the error associated with measured quantities to obtain the error of derived quantities (i.e. functions of measured quantities).

Uncertainty in a Function with Multiple Variables

To propagate error through a function of multiple variables $f(x, y, z, \dots)$ in order to determine the uncertainty of that function (δ_f), one should use the equation:

$$\delta_f = \sqrt{\left(\delta_x \frac{\partial f}{\partial x}\right)^2 + \left(\delta_y \frac{\partial f}{\partial y}\right)^2 + \left(\delta_z \frac{\partial f}{\partial z}\right)^2 + \dots} \quad (1)$$

where x , y , and z are independent variables and δ_x , δ_y , and δ_z are their associated, uncorrelated uncertainties. (Note that in this context, x , y , and z are only variables, and not necessarily associated with cartesian coordinates!) The partial derivatives $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \dots\right)$ quantify the rate of change of the function f with respect to the independent variables x , y , and z .

Usually, the variables x , y , and z represent independent measured quantities, so we will interpret terms like $\left(\delta_x \frac{\partial f}{\partial x}\right)$ as the contribution to the uncertainty in f due to the uncertainty in the measured quantity x alone.

Since f is a function of multiple variables, we can calculate its uncertainty δf by taking the quadratic sum of the uncertainty contribution due to each measured quantity x , y , and z as in eqn (1).

Example:

Suppose we want to calculate the area of a rectangle after measuring the side lengths to be:

$$x = 10.0 \text{ m} \pm .5 \text{ m}$$

$$y = 30.0 \text{ m} \pm .5 \text{ m}$$

The area A of the rectangle will be:

$$A = xy \text{ m}^2 \pm \delta_A \text{ m}^2$$

$$A = 300 \text{ m}^2 \pm \delta_A \text{ m}^2$$

So, to calculate the uncertainty of the area, we will use:

$$\delta_f = \sqrt{\left(\delta_x \frac{\partial f}{\partial x}\right)^2 + \left(\delta_y \frac{\partial f}{\partial y}\right)^2 + \left(\delta_z \frac{\partial f}{\partial z}\right)^2 + \dots}$$

For this particular example:

$$f \equiv A = xy$$

So:

$$\delta_A = \sqrt{\left(\delta_x \frac{\partial A}{\partial x}\right)^2 + \left(\delta_y \frac{\partial A}{\partial y}\right)^2}$$

First calculate the partial derivatives:

$$\frac{\partial A}{\partial x} = \frac{\partial(xy)}{\partial x} = y = 30.0 \text{ m}$$

$$\frac{\partial A}{\partial y} = \frac{\partial(xy)}{\partial y} = x = 10.0 \text{ m}$$

Plug these values, along with δ_x and δ_y (which are both $.5 \text{ m}$) into our uncertainty equation:

$$\delta_A = \sqrt{((.5 \text{ m})(30 \text{ m}))^2 + ((.5 \text{ m})(10 \text{ m}))^2}$$

And we can calculate the uncertainty in the Area, δ_A :

$$\delta_A = 16 \text{ m}^2$$

(Because our measurements are so large (tens of meters), I will keep two sig figs.)

We can now report our calculated area:

$$A = 300 \text{ m}^2 \pm 16 \text{ m}^2$$

Special Cases

Equation 1 can be used to derive uncertainty equations for regularly used common functions. Several of these are:

Error Propagation (Addition and Subtraction)

$$\text{Given: } (A \pm \delta_A) + (B \pm \delta_B)$$

or

$$\text{Given: } (A \pm \delta_A) - (B \pm \delta_B)$$

then:

$$\delta_{total} = \pm \sqrt{\delta_A^2 + \delta_B^2}$$

Equation 2

Error Propagation (Multiplication)

$$\text{Given: } (A \pm \delta_A) \cdot (B \pm \delta_B)$$

then:

$$\delta_{total} = \pm AB \sqrt{\left(\frac{\delta_A}{A}\right)^2 + \left(\frac{\delta_B}{B}\right)^2}$$

Equation 3

Error Propagation (Division)

$$\text{Given: } (A \pm \delta_A)/(B \pm \delta_B)$$

then:

$$\delta_{total} = \pm \frac{A}{B} \sqrt{\left(\frac{\delta_A}{A}\right)^2 + \left(\frac{\delta_B}{B}\right)^2}$$

Equation 4

Density

The density (ρ) of a substance is defined as its mass (m) per unit volume (V).

$$\rho = \frac{m}{V}$$

The density of any pure substance will remain the same regardless of a particular sample's size or mass. Although the S.I. unit for density is kg/m^3 , we will use the unit g/cm^3 . The following table contains densities for substances that you may encounter in this lab.

Material	Density (g/cm³)
Solids	
Metal:	
Aluminum	2.70
Stainless Steel	7.8
Brass	8.44 - 8.75
Bronze	8.74 - 8.89
Copper	8.96
Lead	11.3
Mercury	13.5336
Rock:	
Granite	2.64 - 2.76
Slate	2.6 - 3.3
Diamond	3.51
Garnet	3.15 - 4.3
Corundum	3.9 - 4.0
Wood:	
Pine (Yellow)	0.37 - 0.60
Oak	0.60 - 0.90
Ebony	1.11 - 1.33
Misc.:	
Ice	0.917
Bone	1.7 - 2.0
Chalk	1.9 - 2.8
Glass (Lead)	3 - 4
Fluids	
Atmosphere (STP)	0.001225
Water (20°C)	0.99821
Water (0°C)	0.99984
Mercury (20°C)	13.546

Table 1.1: Density of Selected Materials

PROCEDURE

Part 1: Determining Density of an irregularly shaped object.

- 1) Your TA will show you an irregularly shaped object. Measure the mass of the object with a digital balance and record it in Table 6.
- 2) Determine the volume of the object by using the Vernier Caliper. Note that there are three regularly shaped components to the object: a rectangular prism (*Volume 1*), a protruding cylinder (*Volume 2*), and cylindrical indentation (*Volume 3*). These three volumes must be combined in order to determine the total volume. Record your measurements in Table 5.

You will need to use the outside jaws, inside jaws, and depth gauge on the Vernier Caliper to make your measurements. Be as accurate as possible! Record the total volume in Table 6.

- 3) Calculate the density of the object. Record this density in Table 6.
- 4) Determine the uncertainty in the Volume using Equation 1. Include this with your calculated volume in Table 6. (You will need to take several partial derivatives, which should be recorded in Table 7.)
- 5) Determine the uncertainty in the density using Equation 4. Include this with your calculated volume in Table 6.

Part 2: Improving Last Week's Uncertainty

- 6) Use Equation 1 to recalculate the uncertainty in the density of the metal cylinder that you measured last week. (You will need to recalculate the uncertainty in volume as part of this exercise.)

How does this new uncertainty compare to the uncertainty from the "half max-min" method?