Supplementary Material: Multivariable Calculus

This will be a short, self-contained introduction to the calculus of multiple variables with a focus on calculating partial derivatives. We will only outline the important points that are relevant for lab work calculations, so we refer the reader to any introductory calculus textbook for a more complete treatment (e.g. Stewart's *Essential Calculus*).

Single Variable Differentiation

In single-variable calculus, we consider functions $f(x)$ that depend only on one independent variable. We learn that the rate of change of the function $f(x)$ is given by the derivative

$$\left.\frac{df}{dx}\right|_{x=a} = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\ . \tag{1}$$

The derivative describes the rate of change of a function $f(x)$ as the variable x is allowed to vary. Many useful physical quantities are described by derivatives. For instance, the first time-derivative of position $x(t)$ is known as the velocity $\frac{dx}{dt} = v(t)$ which describes the rate at which the position of an object changes as a function of time.

The limit in equation (1) is necessary for a rigorous definition of the derivative, but once we know how to differentiate some simple functions, we can use simpler rules to calculate derivatives without using the limit definition every time. Some examples of typical derivatives include

$$\begin{aligned}
&\frac{d}{dx}\,x = 1 && \frac{d}{dx}\cos x = \sin x \\
&\frac{d}{dx}x^2 = 2x && \frac{d}{dx}\tan x = \sec^2 x \\
&\frac{d}{dx}\,x^n = n\,x^{n-1} && \frac{d}{dx}e^x = e^x \\
&\frac{d}{dx}\sin x = \cos x && \frac{d}{dx}\,a^x = a^x\ln a.
\end{aligned} \tag{2}$$

Suppose we would like to consider the motion of an object whose position is given by the equation $y(t) = y_0 + v_0 t + \frac{1}{2}a\,t^2$. This expression for the position $y(t)$ describes the one-dimensional motion of an object that experiences a constant acceleration a. We can see from the equation for $y(t)$ that the object must be in motion since its position is changing with time.

We can learn more about the motion of the object by taking time derivatives of the position $y(t)$. The velocity $v(t)$ of the object at any time t is obtained by taking a time derivative of the position $y(t)$

$$\frac{dy}{dt} = \frac{d}{dt}\left(y_0 + v_0 t + \frac{1}{2}at^2\right) = \ v_0 + at$$

where we have used the derivative rules above to carry out the differentiation.

The resulting expression for the velocity $v(t) = v_0 + at$ describes the rate of change of the position as time t is allowed to vary. If we take another time derivative of the position $y(t)$, we obtain the acceleration of the object

$$\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{d}{dt}(v_0 + at) = a.$$

The acceleration describes the rate at which the velocity changes with time. We notice that the expression for the acceleration as a function of time $a(t) = a$ is a constant function for all time t (as it must be since we began with a position equation $y(t)$ that describes the motion of an object undergoing constant acceleration).

Multivariable Functions and Partial Differentiation

It is possible to generalize derivatives of single-variable functions to derivatives of functions of multiple independent variables. Suppose we consider a function $f(x, y)$ of two independent variables x and y

$$f(x, y) = x^2y - 2xy\,. \tag{2}$$

The function $f(x, y)$ takes in two inputs (the independent variables x and y) and returns a single number as an output $f(x, y)$. For example,

$$f(2,3) = (2)^2(3) - 2(2)(3) = 0.$$

When we describe the variables x and y as independent variables, we mean that we have the freedom to change one of the variables, say x, without affecting the value of the other variable y.

To define the derivative of a function of multiple variables, we must consider the fact that the independence of the input variables means that the function f changes in different ways as the independent input variables are allowed to vary.

We can generalize the concept of a derivative of single-variable functions to multivariable functions with a *partial derivative.* As we will see, the partial derivative provides us with a way to quantify how the multivariable function f changes as one of the input variables is allowed to vary.

Consider a function $f(x, y)$ of two independent variables x and y. By analogy with the definition equation (1), we can define the ***partial derivative of f(x,y) with respect to*** $\boldsymbol{x}$ as

$$\frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x + h, y) - f(x, y)}{h}\,. \tag{3}$$

Likewise, we define the ***partial derivative of*** $\boldsymbol{f(x, y)}$ ***with respect to y*** as

$$\frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x, y + h) - f(x, y)}{h}\,. \tag{4}$$

As with the definition of the derivative in equation (1), we will find it more convenient to work with the rules of partial differentiation by taking some partial derivatives rather than appealing to the limit definitions equations (3) and (4).

Examples

To take a partial derivative of a multivariable function $f(x, y)$, you must first identify the variable that you would like to use to take the derivative with respect to, say x. Then apply the derivative rules to the function while *treating all other independent variables as constant.*

The resulting expression (the partial derivative of f) describes the rate of change of the function $f(x, y)$ as the variable x is allowed to change while y is held constant.

Example 1

Consider the multivariable function $f(x, y) = x^2 y$ that has two independent variables x and y. Suppose that we would like to find the partial derivative of $f(x, y)$ with respect to the independent variable x. We have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\,(x^2 y) = 2xy. \tag{5}$$

Since we are differentiating with respect to the variable x, we treat the independent variable y as a constant, so it remains unaffected by the partial derivative. We can also take the partial derivative of $f(x, y)$ with respect to y,

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\,(x^2 y) = x^2. \tag{6}$$

Now the independent variable x is treated as a constant, and it remains unaffected by the partial differentiation with respect to y.

Example 2

Let the multivariable function $g(x, y, z)$ be

$$g(x, y, z) \; = xy - yz^2 + 3xyz.$$

Taking the partial derivative of $g(x, y, z)$ with respect to each of the three independent variables x, y, and z, we obtain

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}(xy - yz^2 + 3xyz) = y + 3yz$$

$$\frac{\partial g}{\partial y} = \frac{\partial}{\partial y}(xy - yz^2 + 3xyz) = x - z^2 + 3xz$$

$$\frac{\partial g}{\partial z} = \frac{\partial}{\partial z}(xy - yz^2 + 3xyz) = -2yz + 3xy.$$