In class we considered a planar quantum rotor model with a symmetry-breaking term that favours angular orientation near $\phi = 0$:

$$\hat{H} = \frac{\hat{L}^2}{2I} + \Omega^2 \cos \hat{\phi}.$$ 

Here, $I$ is the moment of inertia, and $\Omega$ is the natural frequency of the corresponding classical problem in the small-amplitude-oscillation limit. The operators $\hat{\phi}$ and $\hat{L}$ obey the canonical commutation relationship $[\hat{\phi}, \hat{L}] = i\hbar$. We made the decision to work in the $\phi$-representation, so that the operators take the form $\phi$ and $L = (\hbar / i) \partial / \partial \phi$ and act on a wave function $\psi(\phi)$.

1. (2 point) Show that states $\chi_m(\phi) \sim \exp(im\phi)$ are eigenstates of the angular momentum operator with eigenvalue $\hbar m$. Determine the proper normalization of the states.

2. (2 point) Argue that the parity operation (reflection across the preferred axis, $\phi \rightarrow -\phi$) is a symmetry of the Hamiltonian. Construct a basis of states of definite even ($P = +1$) and odd ($P = -1$) parity from linear combinations of the angular momentum states $\chi_m(\phi)$. Explain how this basis can be used to block diagonalize the Hamiltonian.

3. (4 points) Consider a truncated basis that contains only the two lowest-lying states in each of the even- and odd-parity sectors. Write the time-independent Schrödinger equation as two $2 \times 2$ matrix eigenvector problems.

4. (4 points) Solve the even-parity $2 \times 2$ eigenproblem. For the ground state, plot the probability density $|\psi(\phi)|^2$ of finding the rotor in the vicinity angle $\phi$. Do this for small, intermediate, and large values of $\Omega$.

5. (3 points) Compute the expectation values of $\hat{H}$ with respect to the next-lowest-lying states in each sector. Based on energy comparisons, comment on the appropriateness of the basis truncation.