Numerical Integration
- Background
- Trapezoid
- Simpson’s 1/3 rule
- Other methods: Monte Carlo

CLASS NOTES
- HW#4 due today.
- You should be reading in Appendix A.3
Integration is the inverse operation to differentiation. The integral of a function is a measure of the “area” bounded by the function and the horizontal axis. Many analytic functions can be integrated analytically to get a closed form solution. Area above the axis is (+), and area below the axis is (-).

\[ \int_{a}^{b} f(x) \, dx = \text{area} \]
SOME EXACT SOLUTIONS

\[ \int x^m \, dx = \frac{1}{m+1} x^{m+1} \]

\[ \int \frac{1}{x} \, dx = \ln(x) \]

\[ \int \sin(x) \, dx = -\cos(x) \]

\[ \int_0^\pi \sin(x) \, dx = 2.00 \]

\[ \int_a^b x^m \, dx = \left[ \frac{1}{m+1} x^{m+1} \right]_a^b \]
On a computer, integration becomes summation. Numerical integration is also called *quadrature*. Break integral range into multiple “panels” with a simple shape. Add areas of all panels to get total area. Newton-Cotes Method: top of each panel defined by a polynomial of order $n$.

- $n=1$: line - Trapeziod method
- $n=2$: parabola – Simpson’s $1/3$
- $n=3$: cubic – Simpson’s $3/8$

For many applications, Trapezoid method is good enough.
TRAPEZOID METHOD

- Geometry of panels is trapezoidal.
- Area of a trapezoid:
  \[ [f(a+h) + f(a)] \frac{h}{2} \]
- Total area for N panels:
  \[
  I = I_1 + I_2 + I_3 + \ldots + I_N \\
  = \sum_{i=1}^{N-1} [f(a + ih) + f(a + (i + 1)h)] \frac{h}{2}
  \]
def trap(f,xmin,xmax,numPanels):
    Isum=0.
    h=(xmax-xmin)/float(numPanels)
    for i in range(numPanels):
        x=xmin+i*h
        Isum+=(f(x+h)+f(x))*h/2.
    return Isum

- Trapezoid error is of order $\sim h^2$
- Trapezoid method is can be coupled with Richardson extrapolation to improve error further
  + known as the Romberg method
**SIMPSON’S 1/3 RULE**

- Panel cap is a quadratic curve rather than a straight line.

\[ \int_{a}^{b} f(x) \, dx \approx \left[ f(a) + 4f \left( \frac{a + b}{2} \right) + f(b) \right] \frac{h}{3} \]

- Requires evaluation at 3 points rather than 2.
- Error is of order \( \sim h^4 \).
def simp13(f, xmin, xmax, numPanels):
    Isum = 0.
    h = (xmax - xmin) / float(numPanels)
    for i in range(0, numPanels, 2):
        x = xmin + i * h
        Isum += (f(x) + 4 * f(x + h) + f(x + 2 * h)) * h / 3
    return Isum
EXERCISE: TRAPEZOID ERROR

- Use the trapezoid method to estimate the integral:

$$\int_{0}^{1} e^{-x} \, dx = -[e^{-1} - e^{0}] = 0.632120558829$$

- Compute and plot the % error vs number of panels.