Chapter 5 Magnetostatics

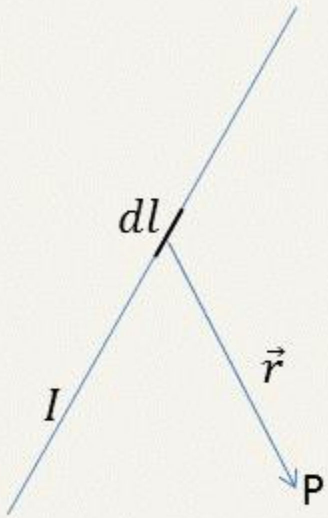
5.1 The Lorentz Force Law

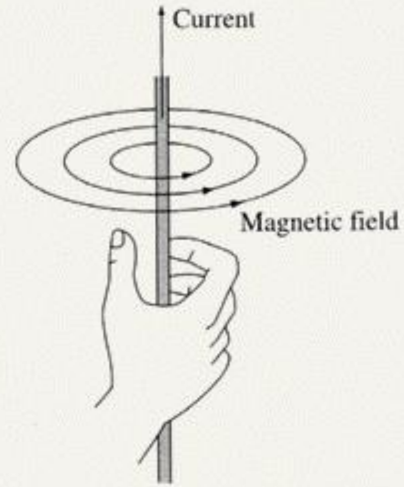
5.2 The Biot-Savart Law

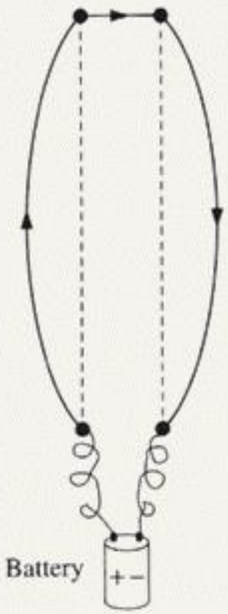
5.3 The Divergence and Curl of $\vec{B}$

5.4 Magnetic Vector Potential

5.1.2 Magnetic Force







(a) Currents in opposite directions repel.

Moving charges (currents) produce magnetic fields,

Biot-Savart Law: for a steady line current

$$\vec{B}(p) = \frac{\mu_0}{4\pi}\int \frac{\vec{I}\times\hat{r}}{r^2}dl$$

$$= \frac{\mu_0}{4\pi} I \int \frac{d\vec{l}\times\hat{r}}{r^2}$$

$tesla\,(T)$

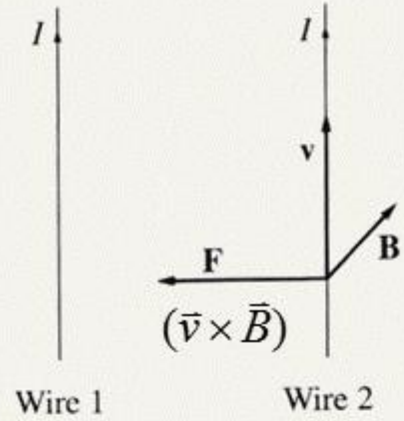
$1\,tesla = 1\,\frac{N}{A\cdot m}$

$cgs : gauss$

$1\,T = 10^4\,gauss$

$\mu_0 = 4\pi\times 10^{-7}\,\frac{N}{A^2}$ Permeability of free space

5.1.2 Magnetic Force

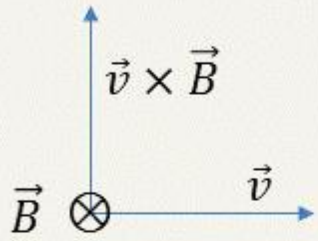


A particle moving in a magnetic field is subjected to a magnetic force

$$\vec{F}_{mag} = Q(\vec{v}\times\vec{B})$$ Lorentz Force Law

Magnetic forces do not work for *a* moving charge Q $d\vec{l} = \vec{v}dt$

$$dW_{mag} = \vec{F}_{mag}\cdot d\vec{l}$$
$$= Q(\vec{v}\times\vec{B})\cdot\vec{v}dt$$
$$= 0$$

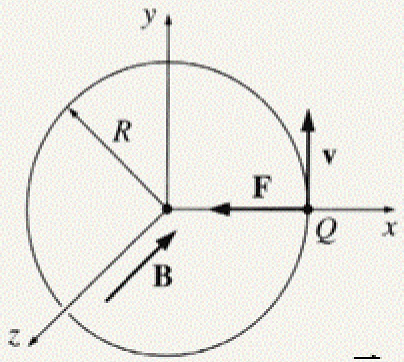


When both electrical and magnetic fields present:

$$\vec{F} = Q[\vec{E} + (\vec{v}\times\vec{B})]$$

Example: Motion of a charge particle perpendicular to magnetic field (cyclotron motion)

Since the magnetic force is always perpendicular to the direction of motion, it moves in a circular path.



$$\vec{F}_{mag} = QvB$$

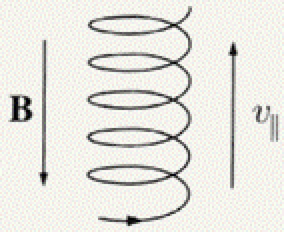
This acts as the centripetal force

$$\vec{F}_{centripetal} = m\frac{v^2}{R} = QvB \quad \Rightarrow \quad mv.\frac{v}{R} = QvB$$

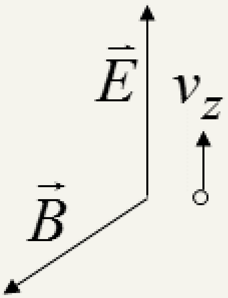
momentum $p = mv = QBR$

period $\dfrac{2\pi R}{v} = \dfrac{m}{QB}$ frequency $w_c = \dfrac{QB}{m}$ cyclotron frequency

If the motion is not perpendicular, parallel component of the motion is not affected, so the particle moves in a spiral.



Example: When there is an electric field perpendicular to magnetic field, and if a particle is moving.



$$\vec{F}_B = q v_z B \hat{y}$$

Suppose particle is released at rest at the origin $\quad v(t=0)=0$

$$m\frac{d\vec{v}}{dt} = q(\vec{E} + \vec{v}\times\vec{B}) \quad v_y = \frac{dy}{dt} \quad v_z = \frac{dz}{dt} \qquad \vec{E} = E\hat{z} \quad \vec{B} = B\hat{x}$$

$$m\frac{d^2y}{dt^2} = q\frac{dz}{dt}B \quad m\frac{d^2z}{dt^2} = q(E - \frac{dy}{dt}B)$$

$$\frac{d^2y}{dt^2} = \omega\frac{dz}{dt} \quad ; \quad \frac{d^2z}{dt^2} = \omega(\frac{E}{B} - \frac{dy}{dt}) \qquad \omega = \frac{qB}{m}$$

$$\frac{d^2y}{dt^2} = \omega \frac{dz}{dt} \quad ; \quad \frac{d^2z}{dt^2} = \omega\left(\frac{E}{B} - \frac{dy}{dt}\right) \qquad \omega = \frac{qB}{m}$$

$$y(t) = C_1 \cos \omega t + C_2 \sin \omega t + \frac{E}{B} t + C_3$$

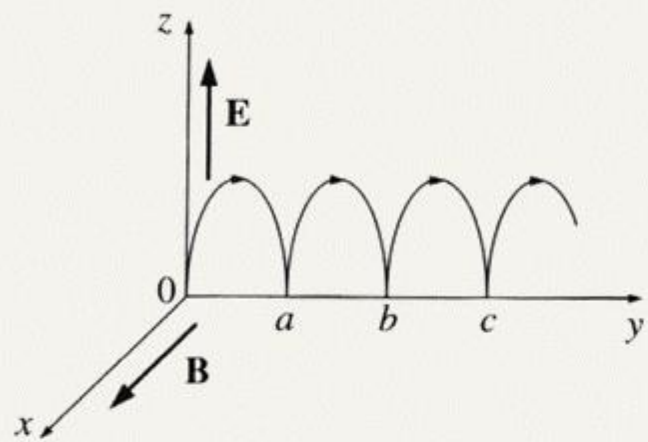
$$z(t) = C_1 \cos \omega t + C_2 \sin \omega t + C_4$$

$$y(0) = z(0) = 0 \; ; \quad \dot{y}(0) = \dot{z}(0) = 0$$

Initial conditions

$$y(t) = \frac{E}{\omega B}(\omega t - \sin \omega\, t)$$

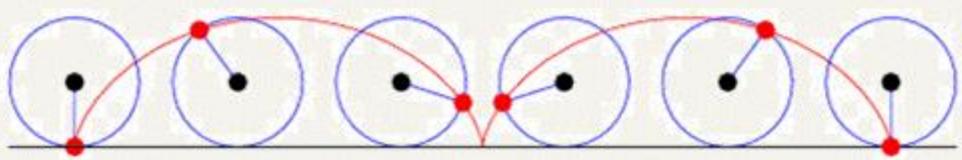
$$z(t) = \frac{E}{\omega B}(1 - \cos \omega\, t)$$



$$\left(y - R\omega t\right)^2 + \left(z - R\right)^2 = R^2 \qquad R = \frac{E}{\omega B}$$

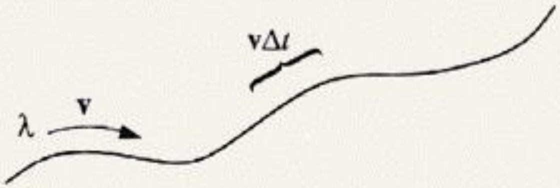
A circle of radius R whose center $(0, \omega t, R)$ moving in y at a speed : Cycloid Motion

$$\omega R = \frac{E}{B}$$



5.1.3 Currents

The current in a wire is the charge per unit time passing a given point.



If the line charge density is λ, and the speed of charges is v

$$\left(I = \frac{(\lambda v \Delta t)}{\Delta t} = \lambda v \quad \Rightarrow \vec{I} = \lambda \vec{v} \right)$$

Current is measured in Amperes 1A = 1 C/S

The magnetic force on a segment of current-carrying wire

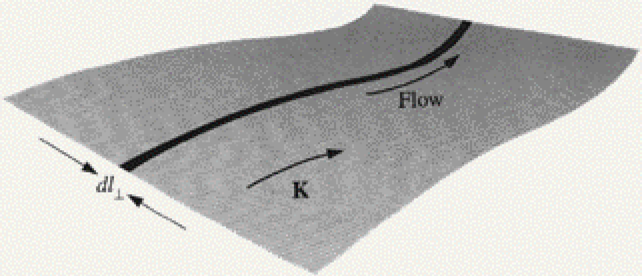
$$\vec{F}_{mag} = \int (\lambda\, dl)\left(\vec{v} \times \vec{B}\right) = \int \left(\vec{I} \times \vec{B}\right) dl$$

$$= \int I\left(d\vec{l} \times \vec{B}\right) \qquad \vec{I} = Id\hat{l}$$

$$\vec{F}_{mag} = I\int \left(d\vec{l} \times \vec{B}\right)$$

Surface and Volume currents

When charges flows over a surface it is described by a surface current density



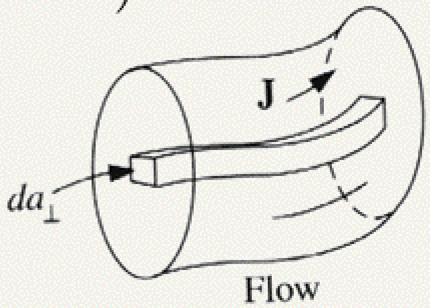
$$\vec{K} = \frac{d\vec{I}}{dl_\perp}$$

the current per unit length-perpendicular-to-flow

$$\vec{K} = \sigma\,\vec{v} \qquad \sigma : surface\ charge\ density$$

The magnetic force on a surface current is

$$\vec{F}_{mag} = \int (\sigma\, da)\left(\vec{v}\times\vec{B}\right) = \int \left(\vec{K}\times\vec{B}\right) da$$



Similarly when the charges distributed throughout space it is described by a volume current density

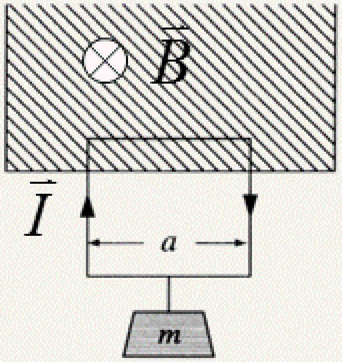
volume current density, The current per unit area-perpendicular-to-flow

$$\vec{J} = \frac{d\vec{I}}{da_\perp} = \rho\,\vec{v}$$

The magnetic force on a volume current is

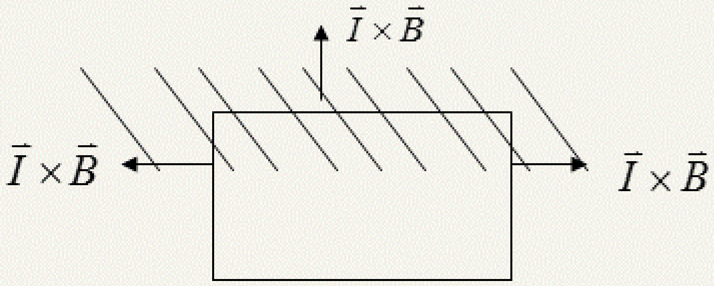
$$\vec{F}_{mag} = \int (\rho\, d\tau)\left(\vec{v}\times\vec{B}\right) = \int \left(\vec{J}\times\vec{B}\right) d\tau$$

Example: A mass *m* is hanging from a rectangular loop with one end in a uniform magnetic field B, what is the value if current needed to support *m* .

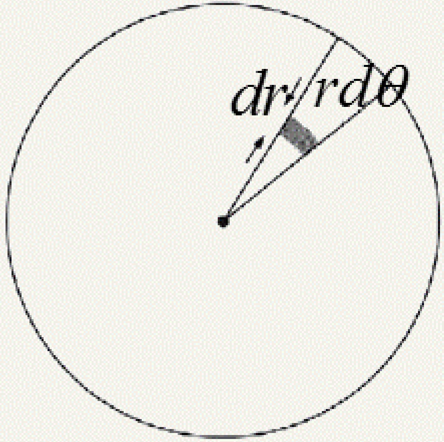


$$F_{mag} = IBa = mg$$

$$I = \frac{mg}{Ba}$$



The current density in circular wire of radius a is given by $J = kr$, find the total current in the wire.



$$I = \int Jda = \int (kr)(rdrd\theta)$$

$$= 2\pi k \int_0^R r^2 dr = \frac{2\pi kR^3}{3}$$

Continuity equation

Since the total current crossing a surface S : $I = \int_s J da_\perp = \int_s \vec{J} \cdot d\vec{a}$

total charge leaving a volume V is $\oint_s \vec{J} \cdot d\vec{a} = \int_V \left(\nabla \cdot \vec{J}\right) d\tau$

Due to charge conservation this is equal to the rate of change of total charge in the volume $= -\frac{d}{dt}\int_V \rho \, d\tau = -\int_V \left(\frac{\partial \rho}{\partial t}\right) d\tau$

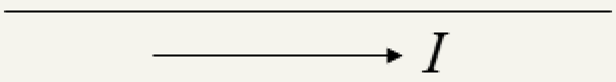
$$\int_V \left(\nabla \cdot \vec{J}\right) d\tau = -\int_V \frac{\partial \rho}{\partial t} d\tau$$

⇒Continuity equation $\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

5.2.1 Steady Currents

Stationary charges ⇒ constant electric field: electrostatics

Steady currents ⇒ constant magnetic field: magnetostatics

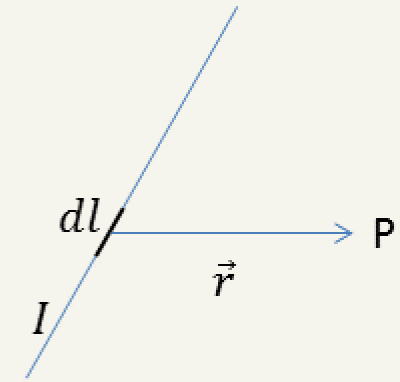


No time dependence

$$\frac{\partial J}{\partial t} = 0, \frac{\partial \rho}{\partial t} = 0 \quad \Rightarrow \quad \nabla \cdot \vec{J} = 0$$

5.2.2 The Magnetic Field of a Steady Current

Moving charges (currents) produce magnetic fields,



Biot-Savart Law: for a steady line current

$$\vec{B}(p) = \frac{\mu_0}{4\pi}\int \frac{\vec{I} \times \hat{r}}{r^2} dl$$

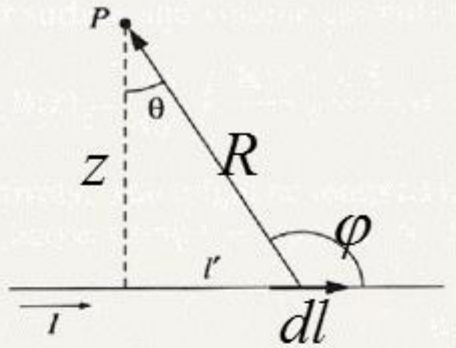
$$= \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times \hat{r}}{r^2}$$

$$\mu_0 = 4\pi \times 10^{-7} \frac{N}{A^2}$$ Permeability of free space

Since $\vec{F}_{mag} = I \times \vec{B} dl$ unit of B is newton per ampere-meter, or tesla

$tesla\ (T) \quad 1\, tesla = 1 \frac{N}{A \cdot m}$ cgs unit: $gauss \quad 1\,T = 10^4\, gauss$

Example: find the magnetic field a distance z from a long straight wire carrying a steady current I



$$d\vec{l} \times \hat{R} = dl \sin\varphi = dl \cos\theta$$

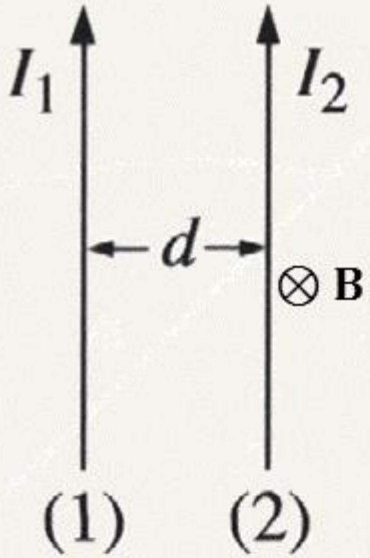
$$l = z\tan\theta \quad dl = z\, d\tan\theta = \frac{z}{\cos^2\theta} d\theta, \; \frac{z}{R} = \cos\theta, \; \frac{1}{R^2} = \frac{\cos^2\theta}{z^2}$$

$$B = \frac{\mu_0}{4\pi} I \int \frac{d\vec{l} \times \hat{R}}{R^2} = \frac{\mu_0}{4\pi} I \int_{\theta_1}^{\theta_2} \left(\frac{\cos^2\theta}{z^2}\right) \cos\theta \frac{z}{\cos^2\theta} d\theta$$

$$= \frac{\mu_0}{4\pi} \frac{I}{z} \int_{\theta_1}^{\theta_2} \cos\theta d\theta = \frac{\mu_0}{4\pi} \frac{I}{z} (\sin\theta_2 - \sin\theta_1)$$

In the case of an infinite wire, $\theta_1 = -\frac{\pi}{2}, \; \theta_2 = \frac{\pi}{2}; \quad B = \frac{\mu_0 I}{2\pi z}$

Force between two current carrying parallel wires:

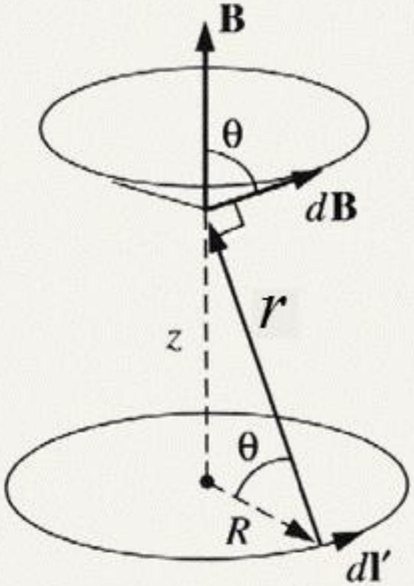


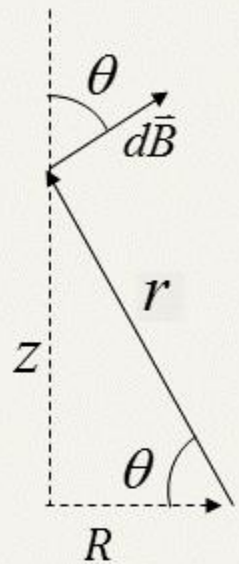
The field at (2) due to I_1 is $\vec{B} = \dfrac{\mu_0 I_1}{2\pi d}$

$$\vec{F} = I_2 \left(\frac{\mu_0 I_1}{2\pi d}\right) \int dl$$

The force per unit length is $f = \dfrac{\mu_0}{2\pi} \dfrac{I_1 I_2}{d}$

Magnetic field from a circular current loop





$$\vec{B} = \int d\vec{B} \cos\theta \,\hat{z}$$

$$= \frac{\mu_0}{4\pi} I \int \frac{dl}{r^2} \cos\theta \,\hat{z}$$

$$= \frac{\mu_0 I}{4\pi} \left(\frac{\cos\theta}{r^2}\right) 2\pi R \hat{z}$$

$$\cos\theta = \frac{R}{r} \qquad r = (R^2 + z^2)^{\frac{1}{2}}$$

$$\vec{B} = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} \hat{z}$$

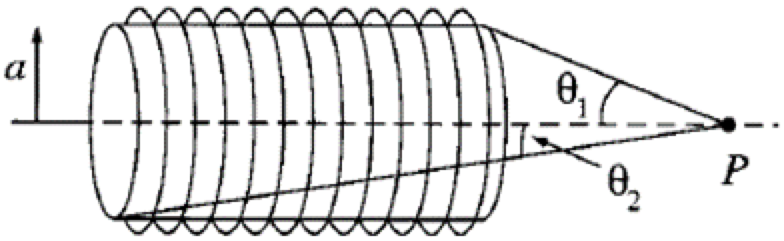


Figure 5.25

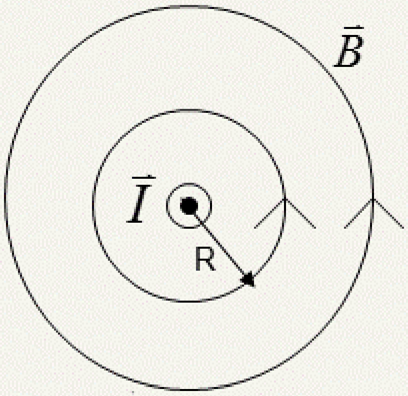
Problem 5.11 Find the magnetic field at point P on the axis of a tightly wound solenoid (helical coil) consisting of n turns per unit length wrapped around a cylindrical tube of radius a and carrying current I (Fig. 5.25). Express your answer in terms of θ_1 and θ_2 (it's easiest that way). Consider the turns to be essentially circular, and use the result of Ex. 5.6. What is the field on the axis of an *infinite* solenoid (infinite in both directions)?

5.3 Curl and Divergence of B

1. Straight line currents:

$$\vec{B} = \frac{\mu_0 I}{2\pi R}$$

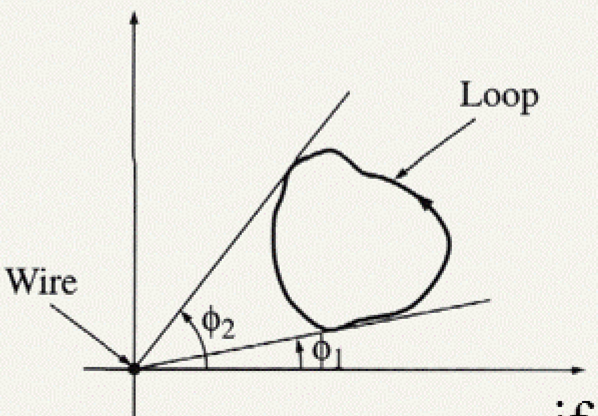
$$\oint \vec{B} \cdot d\vec{l} = \oint \frac{\mu_0 I}{2\pi R} dl = \frac{\mu_0 I}{2\pi R} \oint dl = \mu_0 I$$



For an arbitrary path

$$\vec{B} = \frac{\mu_0 I}{2\pi R}\hat{\varphi} \qquad d\vec{l} = dr\hat{r} + rd\varphi\hat{\varphi} + dz\hat{z}$$

$$\oint \vec{B} \cdot d\vec{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{r} r\, d\varphi = \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\varphi = \mu_0 I$$

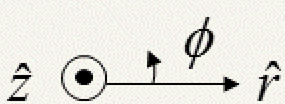


If the loop not enclosing the wire (current)

$$\int_{\varphi_1}^{\varphi_2} d\varphi + \int_{\varphi_2}^{\varphi_1} d\varphi = 0$$

$$\therefore \oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc};$$

if the current is distributed over an area $I_{enc} = \int \vec{J} \cdot d\vec{a}$

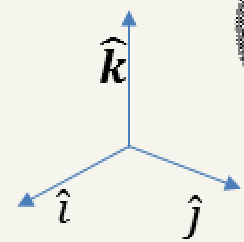


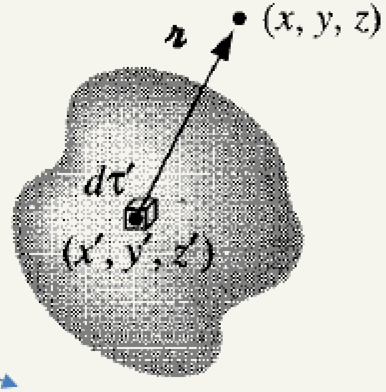
$$\oint \vec{B} \cdot d\vec{l} = \int (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \int \vec{J} \cdot d\vec{a} \Rightarrow \nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

5.3 The Divergence and Curl of B

2. General case:

Biot-savart law for a volume current density





$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$\mathscr{r} = (x-x')\hat{i} + (y-y')\hat{j} + (z-z')\hat{k} \qquad d\tau' = dx'dy'dz'$$

$$B(x,y,z), \quad J(x',y',z')$$

$$\nabla = \hat{i}\partial x + \hat{j}\partial y + \hat{k}\partial z \qquad \nabla' = \hat{i}\partial x' + \hat{j}\partial y' + \hat{k}\partial z'$$

$$\nabla\times\vec{B} = \frac{\mu_0}{4\pi}\int \nabla\times(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2})d\tau'$$

$$\nabla\times(\vec{A}\times\vec{C}) = (\vec{C}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{C} + \vec{A}(\nabla\cdot\vec{C}) - \vec{C}(\nabla\cdot\vec{A})$$

$$\nabla\times(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}) = \underbrace{(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot\nabla)\vec{J}}_{0 \ \because J(x',y',z')} - (\vec{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} + \vec{J}(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}) - \underbrace{\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}(\nabla\cdot\vec{J})}_{0 \ \because J(x',y',z')}$$

$$= \vec{J}\underbrace{(\nabla\cdot\frac{\hat{\mathscr{r}}}{\mathscr{r}^2})}_{4\pi\delta^3(\mathscr{r})} - (\vec{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\nabla\times(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2})=\vec{J}4\pi\delta^3(\mathscr{r})-(\vec{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$$

$$\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}=\frac{x-x'}{\mathscr{r}^3}\hat{i}+\frac{y-y'}{\mathscr{r}^3}\hat{j}+\frac{z-z'}{\mathscr{r}^3}\hat{k}$$

since $\frac{d}{dx}f(x-x')=-\frac{d}{dx'}f(x-x')\Rightarrow(\vec{J}\cdot\nabla)\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}=-(\vec{J}\cdot\nabla')\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}$

since $\nabla\cdot(f\vec{A})=f(\nabla\cdot\vec{A})+\vec{A}\cdot(\nabla f)\Rightarrow\vec{A}\cdot(\nabla f)=\nabla\cdot(f\vec{A})-f(\nabla\cdot\vec{A})$

$$(\vec{J}\cdot\nabla')(\frac{x-x'}{\mathscr{r}^3})=\nabla'\cdot[\frac{x-x'}{\mathscr{r}^3}\vec{J}]-(\frac{x-x'}{\mathscr{r}^3})(\underset{\overset{\parallel}{0}}{\nabla'\cdot\vec{J}})$$

0 for steady currents

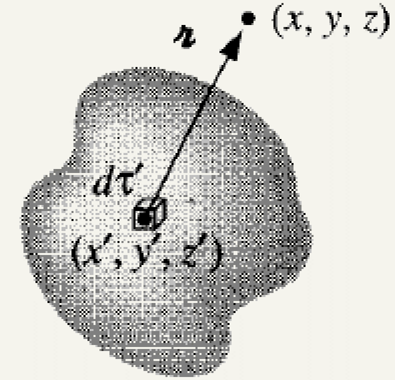
$$\int_{volume}\nabla'\cdot[\frac{x-x'}{\mathscr{r}^3}\vec{J}]d\tau'=\oint_{surface}\frac{x-x'}{\hat{\mathscr{r}}^3}\vec{J}\cdot d\vec{a}\quad\rightarrow 0$$

For a surface far away

$$\nabla\times\vec{B}=\frac{\mu_0}{4\pi}\int\vec{J}(\vec{r}')4\pi\delta^3(\vec{r}-\vec{r}')d\tau'=\mu_0\vec{J}(\vec{r})$$

$\nabla\times\vec{B}=\mu_0\vec{J}(\vec{r})$ Ampere's law in differential form

Divergence of B



$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}d\tau'$$

$$\nabla\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \nabla\cdot(\vec{J}\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2})d\tau'$$

since $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

$$\nabla\cdot(J\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}) = \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\cdot(\nabla\cdot\vec{J}) - \vec{J}\cdot(\nabla\times\frac{\hat{\mathscr{r}}}{\mathscr{r}^2})$$

$\|$ $\quad$ $\|$

$0 \ (\because J(x',y',z'))$ $\quad$ $0 \ (\because \ = \nabla\times\nabla(1/\mathscr{r})$

$B(x,y,z),\ \ J(x',y',z')$

$d\tau' = dx'dy'dz'$

$\nabla = \hat{i}\partial x + \hat{j}\partial y + \hat{z}\partial z$

$\nabla' = \hat{i}\partial x' + \hat{j}\partial y' + \hat{z}\partial z'$

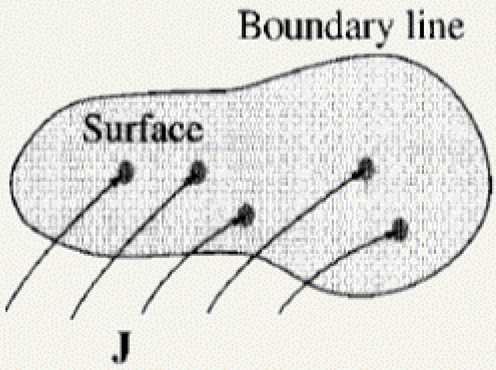
$$\therefore \nabla\times\vec{B} = 0$$

5.3.3 Applications of Ampere's Law

Ampere's Law in differential form

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J}}$$

$$\int (\nabla \times \vec{B}) \cdot d\vec{a} = \oint \vec{B} \cdot d\vec{l} = \mu_0 \int \vec{J} \cdot d\vec{a} = \mu_0 I_{enc}$$



Ampere's Law in integral form

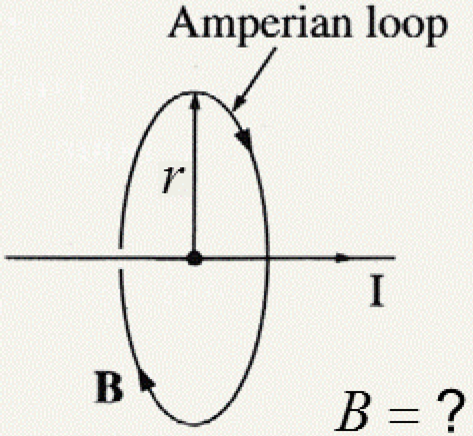
$$\boxed{\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}}$$

Electrostatics:	Coulomb law	Gauss law
	⇕	⇕
Magnetostatics:	Bio-Savart law	Ampere law

Example1: Find the magnetic field a distance r from Long straight wire carrying a current I

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} = \mu_0 I$$

$$B \oint dl = B \cdot 2\pi r = \mu_0 I \quad \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

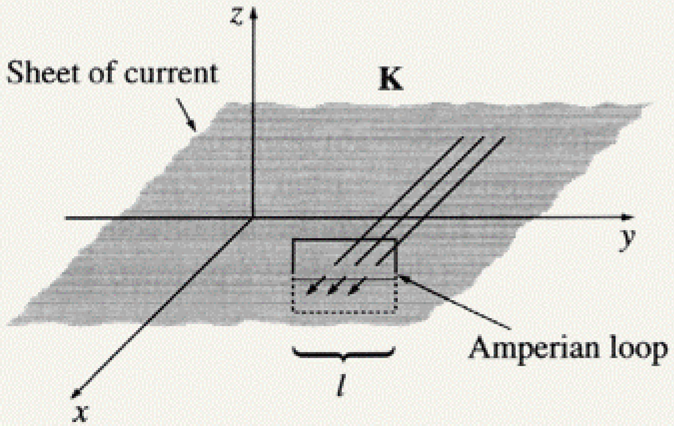


Example 2: Find the magnetic field of a surface current density K on a plane of infinite extent

$$B(z)[\hat{j}]$$

$$\underset{\substack{\| \\ 2Bl}}{\oint B \cdot dl} = \mu_0 I_{enc} = \mu_0 K l$$

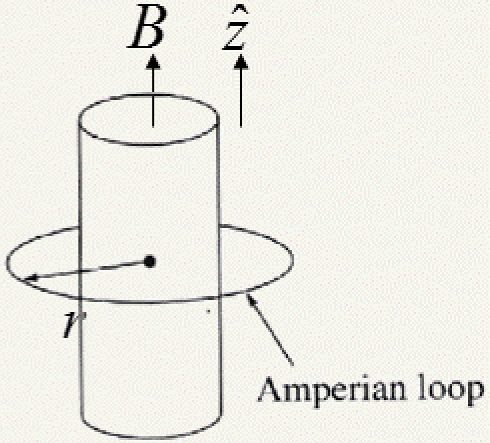
$$\vec{B} = \begin{cases} \frac{\mu_0}{2} K\hat{y} \quad for \quad z < 0 \\ -\frac{\mu_0}{2} K\hat{y} \quad for \quad z > 0 \end{cases}$$



Example: Magnetic field of a long solenoid with n turns per unit length

There cannot be a radial component B_r

Otherwise when the current reserved B_r direction reserves, but reversing current is same as turning solenoid upside down, which should not affect the direction of B_r. So the field is along the axial direction

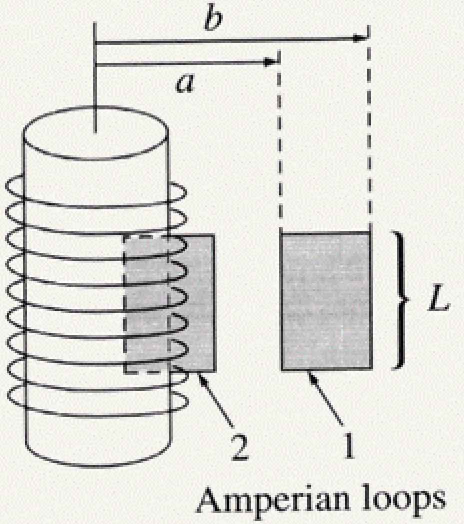


There cannot be a tangential component B_φ

$$\oint \vec{B}\cdot d\vec{l} = B_\phi . 2\pi r = \mu_0 I_{enc} = 0 \quad \Rightarrow B_\phi = 0$$

loop 1.

$$\oint_1 \vec{B}\cdot d\vec{l} = [B(a) - B(b)]L = \mu_0 I_{enc} = 0 \quad \Rightarrow B(a) = B(b)$$



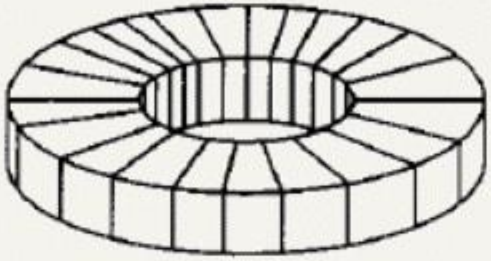
But the field →0 for large r so field outside the solenoid is zero

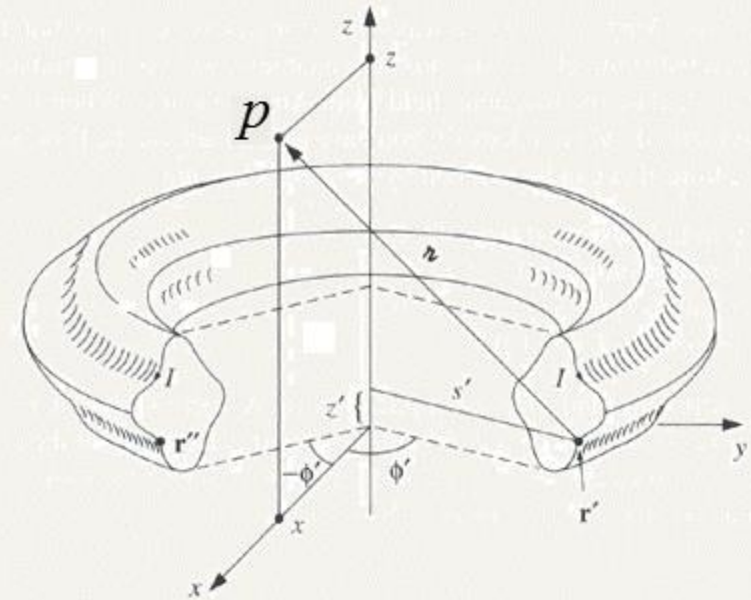
loop 2.

$$\oint_2 \vec{B}\cdot d\vec{l} = BL = \mu_0 I_{enc} = \mu_0 NIL$$

$$B = \begin{cases} \mu_0 NI\hat{z} & inside \\ 0 & outside \end{cases}$$

Magnetic field of a toroidal coil:





Solution:

$$\oint \vec{B}\cdot d\vec{r} = \mu_0 I_{enc} \qquad B(r) = \begin{cases} \dfrac{\mu_0 n I}{2\pi r}\hat{\varphi} & inside \\ 0 & outside \end{cases}$$

5.3.3 (5)

$$\vec{B}(p) = ? \qquad d\vec{B} = \frac{\mu_0}{4\pi}\frac{\vec{I}\times\vec{R}}{R^3}dl \qquad p = (x_0, 0, z_0)$$

$$r' = (r\cos\varphi, r\sin\varphi, z)$$

$$\vec{R} = \vec{p} - \vec{r}\,' = (x_0 - r\cos\varphi, -r\sin\varphi, z_0 - z)$$

$$\vec{I} = I_r\hat{r} + I_z\hat{z} = (I_r\cos\varphi, I_r\sin\varphi, I_z)$$

$$(\because I_\varphi = 0)$$

$$\vec{I}\times\vec{R} = \varepsilon_{ijk}I_iR_j\hat{k}$$

$$= \hat{x}[I_yR_z - I_zR_y] + \hat{y}[I_zR_x - I_xR_z] + \hat{z}[I_xR_y - I_yR_x]$$

$$= \hat{x}[I_r\sin\varphi(z_0 - z) - I_z(-r\sin\varphi)] + \hat{y}[I_z(x_0 - r\cos\varphi)$$

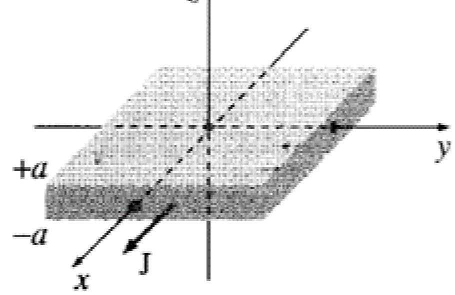
$$-I_r\cos\varphi(z_0 - z)] + \hat{z}[I_r\cos\varphi(-r\sin\varphi) - I_r\sin\varphi(x_0 - r\cos\varphi)]$$

$$= \hat{x}\{\sin\varphi[I_r(z_0 - z) + I_zr]\} + \hat{y}\{I_zx_0 - \cos\varphi[I_zr + I_r(z_0 - z)]\}$$

$$+\hat{z}[-\sin\varphi I_rx_0]$$

$\hat{x}$ and $\hat{z}$ components cancel out $\because$ $\sin\varphi$ from r' and r''

$\therefore \vec{I}\times\vec{R} = (\quad)\hat{y} = (\quad)\hat{\varphi} \qquad \vec{B}$ in $\hat{\varphi}$



Problem 5.14 A thick slab extending from $z = -a$ to $z = +a$ carries a uniform volume current $\mathbf{J} = J\,\hat{\mathbf{x}}$ (Fig. 5.41). Find the magnetic field, as a function of z, both inside and outside the slab.

Problem 5.15 Two long coaxial solenoids each carry current I, but in opposite directions, as shown in Fig. 5.42. The inner solenoid (radius a) has n_1 turns per unit length, and the outer one (radius b) has n_2. Find **B** in each of the three regions: (i) inside the inner solenoid, (ii) between them, and (iii) outside both.

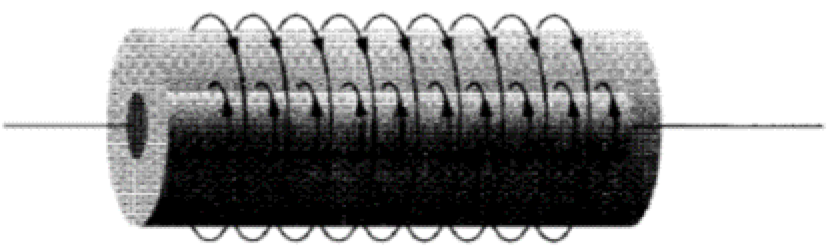


Figure 5.42

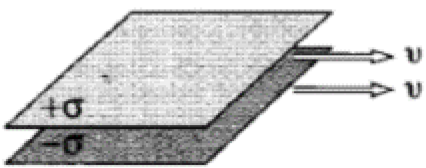


Figure 5.43

Problem 5.16 A large parallel-plate capacitor with uniform surface charge σ on the upper plate and $-\sigma$ on the lower is moving with a constant speed v, as shown in Fig. 5.43.

(a) Find the magnetic field between the plates and also above and below them.

(b) Find the magnetic force per unit area on the upper plate, including its direction.

(c) At what speed v would the magnetic force balance the electrical force?[11]

5.4.1 The Vector Potential

In electrostatics: $\nabla \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} = -\nabla V$

In magnetostatics : since $\nabla \times \vec{B} = \mu_0 I$ there is no U such that $B = \nabla U$

but $\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \nabla \times \vec{A}$

$\vec{A}$ is the vector potential in magnetostatics

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

Electric potential is determined up to a constant

$$\vec{E} = -\nabla V = -\nabla(V + C) = -\nabla V' \qquad V' = V + C$$

Likewise the vector potential has an ambiguity

$\vec{B} = \nabla \times \vec{A} = \nabla \times (\vec{A} + \nabla \lambda)$ λ a scalar (gauge freedom)

$\Rightarrow \vec{B} = \nabla \times \vec{A}' \qquad \vec{A}' = \vec{A} + \nabla \lambda$ (a gauge transformation)

Using this freedom it is always possible to find $\vec{A}$ such that $\nabla . \vec{A} = 0$

(gauge condition, this particular gauge is called the Coulomb gauge)

Suppose $\nabla \cdot \vec{A} \neq 0$

$$\vec{A}' = \vec{A} + \nabla\lambda \quad \Rightarrow \nabla \cdot \vec{A}' = \nabla \cdot \vec{A} + \nabla^2 \lambda$$

Now select λ such that $\nabla^2 \lambda = -(\nabla \cdot \vec{A})$ so $\nabla \cdot \vec{A}' = 0$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$

$$V = \frac{1}{4\pi}\int \frac{\rho}{R} d\tau$$

so $\nabla^2 \lambda = -(\nabla \cdot \vec{A}) \Rightarrow \lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot \vec{A}}{R} d\tau$

So in this (Coulomb) gauge:

$$\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} \Rightarrow \boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

Ampere's Law

$$\boxed{\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{R} d\tau}$$

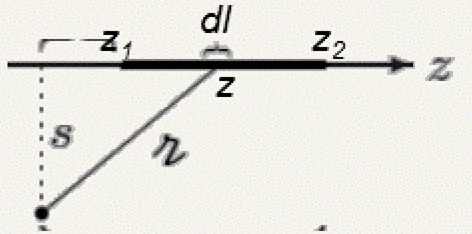
if $\vec{J}(\infty) = 0$

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{R} d\ell = \frac{\mu_0 I}{R}\int \frac{\hat{\ell}}{R} d\ell \, ; \quad \vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{R} da$$

Example: Find the magnetic vector potential of a finite segment of a straight a wire carrying a current I

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{I\,\hat{\mathbf{z}}}{\mathscr{r}}\,dz = \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\int_{z_1}^{z_2} \frac{dz}{\sqrt{z^2+s^2}}$$

$$= \frac{\mu_0 I}{4\pi}\,\hat{\mathbf{z}}\left[\ln\left(z+\sqrt{z^2+s^2}\right)\right]\Big|_{z_1}^{z_2} = \boxed{\frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2+\sqrt{(z_2)^2+s^2}}{z_1+\sqrt{(z_1)^2+s^2}}\right]\hat{\mathbf{z}}}$$



$$\vec{B} = \nabla\times\vec{A} \qquad \nabla\times\mathbf{v} = \left(\frac{1}{s}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{\mathbf{s}} + \left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right)\hat{\boldsymbol{\phi}} + \frac{1}{s}\left[\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial\phi}\right]\hat{\mathbf{z}}$$

$$= \nabla\times\mathbf{A} = -\frac{\partial A}{\partial s}\hat{\phi} = -\frac{\mu_0 I}{4\pi}\left[\frac{1}{z_2+\sqrt{(z_2)^2+s^2}}\frac{s}{\sqrt{(z_2)^2+s^2}} - \frac{1}{z_1+\sqrt{(z_1)^2+s^2}}\frac{s}{\sqrt{(z_1)^2+s^2}}\right]\hat{\phi}$$

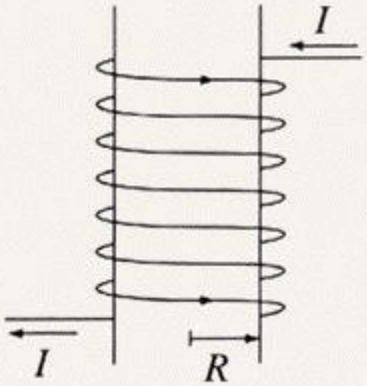
$$= -\frac{\mu_0 I s}{4\pi}\left[\frac{z_2-\sqrt{(z_2)^2+s^2}}{(z_2)^2-[(z_2)^2+s^2]}\frac{1}{\sqrt{(z_2)^2+s^2}} - \frac{z_1-\sqrt{(z_1)^2+s^2}}{z_1^2-[(z_1)^2+s^2]}\frac{1}{\sqrt{(z_1)^2+s^2}}\right]\hat{\phi}$$

$$= -\frac{\mu_0 I s}{4\pi}\left(-\frac{1}{s^2}\right)\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - 1 - \frac{z_1}{\sqrt{(z_1)^2+s^2}} + 1\right]\hat{\phi} = \frac{\mu_0 I}{4\pi s}\left[\frac{z_2}{\sqrt{(z_2)^2+s^2}} - \frac{z_1}{\sqrt{(z_1)^2+s^2}}\right]$$

or, since $\sin\theta_1 = \frac{z_1}{\sqrt{(z_1)^2+s^2}}$ and $\sin\theta_2 = \frac{z_2}{\sqrt{(z_2)^2+s^2}}$,

$$= \boxed{\frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1)\,\hat{\phi}}$$

Example: Vector potential of a solenoid with n turns per unit length



$\vec{A} = ?$

Current extends to infinity so cannot take the integral directly, but

$$\oint \vec{A}\cdot d\vec{\ell} = \int \nabla\times\vec{A}\cdot d\vec{a} = \int \vec{B}\cdot d\vec{a} = \Phi$$

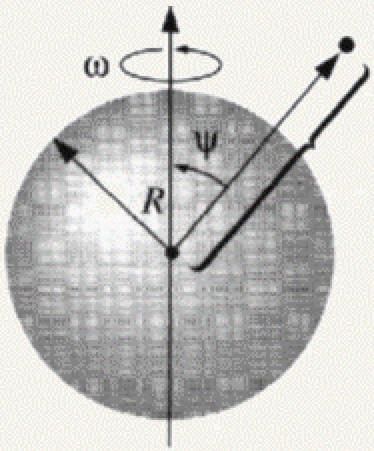
Compare this to ampere law $\vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc}$

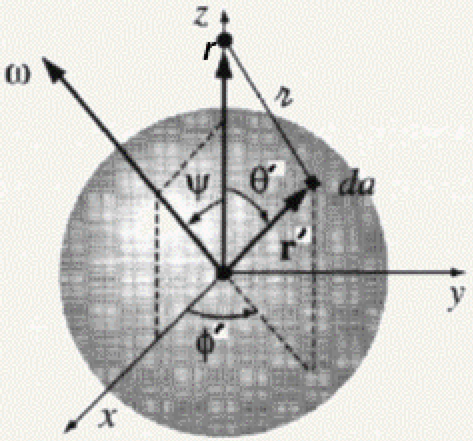
$$\because \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc} \Rightarrow \begin{cases} B_{in} = \mu_0 NI \\ B_{out} = 0 \end{cases}$$

$$\underset{A\,2\pi r}{\underset{\parallel}{\oint \vec{A}\cdot d\vec{\ell}}} = \int \vec{B}\cdot d\vec{a} = \begin{cases} \mu_0 NI(\pi r^2) & r < R \\ \mu_0 NI(\pi R^2) & r > R \end{cases}$$

$$\vec{A} = \begin{cases} \dfrac{\mu_0 NI}{2} r\hat{\varphi} & r < R \\ \dfrac{\mu_0 NI}{2}\dfrac{R^2}{r}\hat{\varphi} & r > R \end{cases}$$

Example: The vector potential of a rotating sphere with uniform surface charge density σ





$$\vec{A}(P) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}}{\mathscr{r}} da$$

surface integration over θ is easier in this orientation of coordinates

$$\vec{K} = \sigma\,\vec{v}, \quad |\mathscr{r}'| = R, \quad \mathscr{r} = (R^2 + r^2 - 2Rr\cos\theta')^{1/2}, \quad da = R^2 \sin\theta'\, d\theta'\, d\varphi'$$

$$\vec{\omega} = \omega\sin\psi\,\hat{x} + \omega\cos\psi\,\hat{z}$$

$$\vec{v} = \vec{\omega}\times\vec{r}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \omega\sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\varphi' & R\sin\theta'\sin\varphi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega[-(\cos\psi'\sin\theta'\sin\varphi')\hat{x} + (\cos\psi\sin\theta'\cos\varphi' - \sin\psi\cos\theta')\hat{y} + (\sin\psi\sin\theta'\sin\varphi')\hat{z}]$$

$$\vec{v} = R\omega[-(\cos\psi' \sin\theta' \sin\varphi')\hat{x} + (\cos\psi \sin\theta' \cos\varphi' - \sin\psi \cos\theta')\hat{y}$$
$$+ (\sin\psi \sin\theta' \sin\varphi')\hat{z}]$$

$$\vec{A}(\vec{r}) = \frac{\mu_0 \sigma}{4\pi} \int \frac{\vec{v}}{\mathscr{r}} R^2 \sin\theta' \, d\theta' \, d\varphi'$$

Since $\int_0^{2\pi} \sin\varphi \, d\varphi = \int_0^{2\pi} \cos\varphi' \, d\varphi' = 0$ and $\mathscr{r}$ is independent of φ'

all terms containg $\sin\varphi'$ or $\cos\varphi'$ integrate to zero

$$\therefore \vec{A}(\vec{r}) = -\left(\frac{\mu_0 R^3 \sigma\omega \sin\psi}{2}\right)\left(\int_0^{\pi} \frac{\cos\theta' \sin\theta'}{(R^2 + r^2 - 2Rr\cos\theta')^{1/2}} d\theta'\right)\hat{y}$$

$$\int_0^{\pi} \frac{\cos\theta' \sin\theta'}{(R^2 + r^2 - 2Rr\cos\theta')^{1/2}} d\theta' = \int_{-1}^{+1} \frac{u}{(R^2 + r^2 - 2Rru)^{1/2}} du$$

$$\begin{pmatrix} u = \cos\theta' \\ du = -\sin\theta' d\theta' \end{pmatrix}$$

$$= \int_{-1}^{+1} \frac{u}{(R^2 + r'^2 - 2Rr'u)^{1/2}} du$$

$$\left(\int \frac{x}{\sqrt{x+a}} dx = \frac{2}{3}(x - 2a)\sqrt{x+a} \right)$$

$$= -\frac{\left(R^2 + r^2 - Rru\right)}{2R^2 r^2} \left(R^2 + r^2 - 2Rru\right)^{1/2} \Bigg|_{-1}^{+1}$$

$$= -\frac{1}{3R^2 r'^2} [|R - r|(R^2 + r^2 + Rr) - (R + r)(R^2 + r^2 - Rr)]$$

if ***r < R***

$$= -\frac{1}{3R^2 r^2} [(R - r)(R^2 + r^2 + Rr) - (R + r)(R^2 + r^2 - Rr)] = \frac{2r}{3R^2}$$

if ***r > R***

$$= -\frac{1}{3R^2 r^2} [(r - R)(R^2 + r^2 + Rr) - (R + r)(R^2 + r^2 - Rr)] = \frac{2R}{3r^2}$$

$$\therefore \vec{A}(\vec{r}) = -\left(\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2} \right) \begin{pmatrix} \dfrac{2r}{3R^2} \\ \dfrac{2R}{3r^2} \end{pmatrix} \hat{y}$$

$$\therefore \vec{A}(\vec{r}) = -\left(\frac{\mu_0 R^3 \sigma\omega \sin\psi}{2}\right)\begin{pmatrix} \dfrac{2r}{3R^2} \\ \dfrac{2R}{3r^2} \end{pmatrix}\hat{y} \qquad \begin{matrix} r < R \\ r > R \end{matrix}$$

since $\vec{\omega}\times\vec{r} = -\omega r \sin\psi \hat{y}$

$$\Rightarrow \vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_0 R\sigma}{3}\vec{\omega}\times\vec{r} \text{ inside sphere} \\ \dfrac{\mu_0 R^4\sigma}{3r^3}\vec{\omega}\times\vec{r} \text{ outside sphere} \end{cases}$$

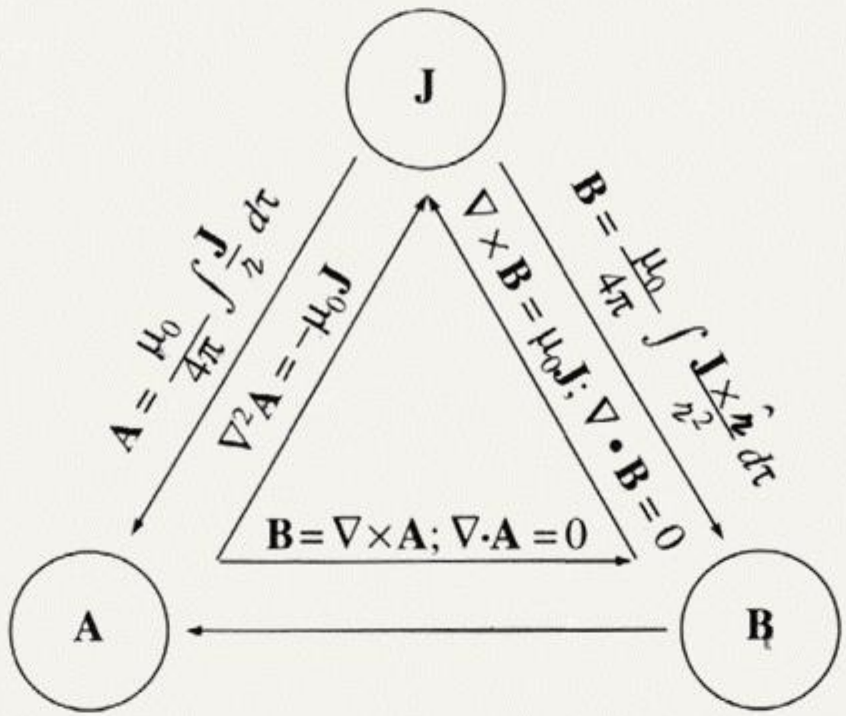
Field inside the sphere $\qquad \vec{B} = \nabla\times\vec{A}(\vec{r}) = \dfrac{2\mu_0 R\sigma}{3}\vec{\omega}$

$$\begin{pmatrix} \nabla\times(\vec{A}\times\vec{B}) = (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B} + \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) \\ \Rightarrow \nabla\times(\vec{\omega}\times\vec{r}) = (\vec{r}\cdot\nabla)\vec{\omega} - (\vec{\omega}\cdot\nabla)\vec{r} + \vec{\omega}(\nabla\cdot\vec{r}) - \vec{r}(\nabla\cdot\vec{\omega}) \\ = \ 0 \qquad -\vec{\omega} \qquad +3\vec{\omega} \qquad -0 \quad = 2\vec{\omega} \end{pmatrix}$$

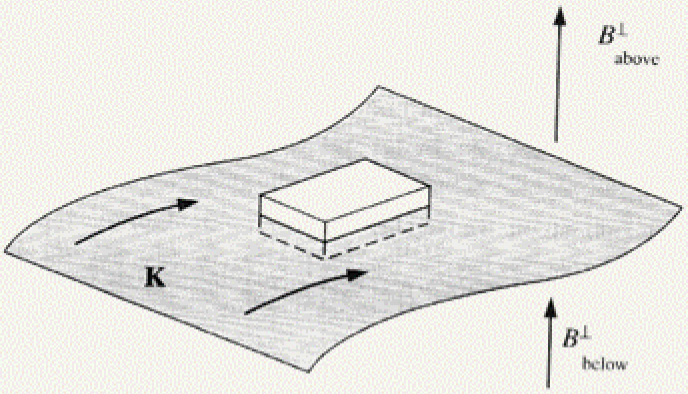
Summary

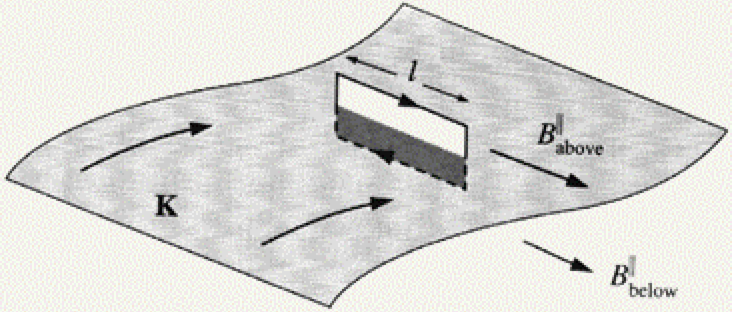
$$\nabla \cdot \vec{B} = 0 \quad \nabla \times \vec{B} = \mu_0 \vec{J} \quad \Rightarrow \quad \vec{B}(p) = \frac{\mu_0}{4\pi} \int \frac{\vec{I} \times \hat{r}}{r^2} d\tau$$

$$\nabla \cdot \vec{A} = 0 \qquad \nabla \times \vec{A} = \vec{B} \qquad \Rightarrow \quad \vec{A} = \frac{1}{4\pi} \int \frac{J}{r^2} d\tau$$



Magnetostatic Boundary Conditions





$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\boxed{B^{\perp}{}_{above} = B^{\perp}{}_{below}}$$

$$\oint \vec{B} \cdot d\vec{\ell} = (B^{\parallel}{}_{above} - B^{\parallel}{}_{below})\ell$$

$$= \mu_0 I_{enc} = \mu_0 K \ell$$

$$\boxed{B^{\parallel}{}_{above} - B^{\parallel}{}_{below} = \mu_0 K}$$

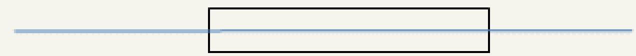
$$\boxed{B_{above} - B_{below} = \mu_0 (K \times \hat{n})}$$

$$(E^{\parallel}{}_{a} = E^{\parallel}{}_{b})$$

$$(E^{\perp}{}_{a} - E^{\perp}{}_{b} = \frac{\sigma}{\varepsilon_0} \hat{n})$$

Boundary conditions for the Vector Potential $\vec{A}$

$$\nabla \cdot \vec{A} = 0 \quad \Rightarrow \quad A^{\perp}{}_{above} = A^{\perp}{}_{below}$$



$\nabla \times \vec{A} = \vec{B} \quad \Rightarrow \quad \oint \vec{A} \cdot d\vec{l} = \int \vec{B} \cdot d\vec{a} = \Phi \rightarrow 0$ for a loop very close to surface

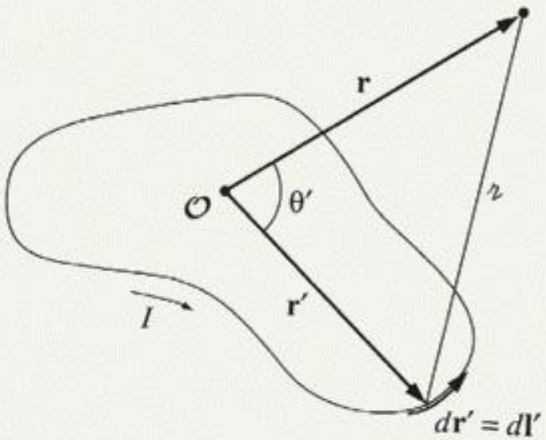
$$A^{\parallel}{}_{above} = A^{\parallel}{}_{below} \quad \Rightarrow \quad \vec{A}_{above} = \vec{A}_{below} \qquad (V_a = V_b)$$

$$(\nabla \times \vec{A})_{above} - (\nabla \times \vec{A})_{below} = \mu_0 (\vec{K} \times \hat{n})$$

$$(\hat{n} \frac{\partial}{\partial n} \times \vec{A})_{above} - (\hat{n} \frac{\partial}{\partial n} \times \vec{A})_{below} = \mu_0 (\vec{K} \times \hat{n})$$

$$\boxed{\frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}} \qquad \frac{\partial V_a}{\partial n} - \frac{\partial V_b}{\partial n} = -\frac{\sigma}{\varepsilon_0}$$

5.4.3 Multipole Expansion of the Vector Potential



$$\vec{A} = \frac{\mu_0 I}{4\pi}\oint \frac{1}{R} d\vec{\ell} \quad \text{(for a line current)}$$

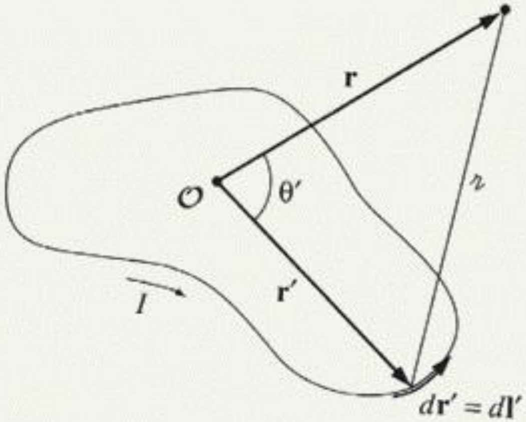
$$\frac{1}{\mathscr{r}} = \frac{1}{(r^2 + r'^2 - 2rr'\cos\theta)^{1/2}} \cong \frac{1}{r[1 - 2(r'/r)\cos\theta']^{1/2}} \cong \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta' + \cdots\right)$$

$$r >> r'$$

$$\vec{A} \cong \frac{\mu_0 I}{4\pi}\oint \frac{1}{r}\left(1 + \frac{r'}{r}\cos\theta' + \cdots\right) d\vec{l} = \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l} + \frac{1}{r^2}\oint r'\cos\theta' d\vec{l} + \cdots\right]$$

$$\frac{\mu_0 I}{4\pi r}\oint d\vec{\ell} = \text{monopole term =0 no magnetic monopoles !}$$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\theta' d\vec{l} = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{r}\cdot\vec{r}')d\vec{l}$$



$$\hat{r}\times\oint(\vec{r}'\times d\vec{r}') = \oint \vec{r}'(\hat{r}\cdot d\vec{r}') - \oint d\vec{r}'(\hat{r}\cdot\vec{r}')$$

$$= \oint d\left[(\vec{r}'(\hat{r}\cdot\vec{r}')\right] - \oint(\hat{r}\cdot\vec{r}')d\vec{r}' - \oint d\vec{r}'(\hat{r}\cdot\vec{r}')$$

0

$$= -2\oint(\hat{r}\cdot\vec{r}')d\vec{r}'$$

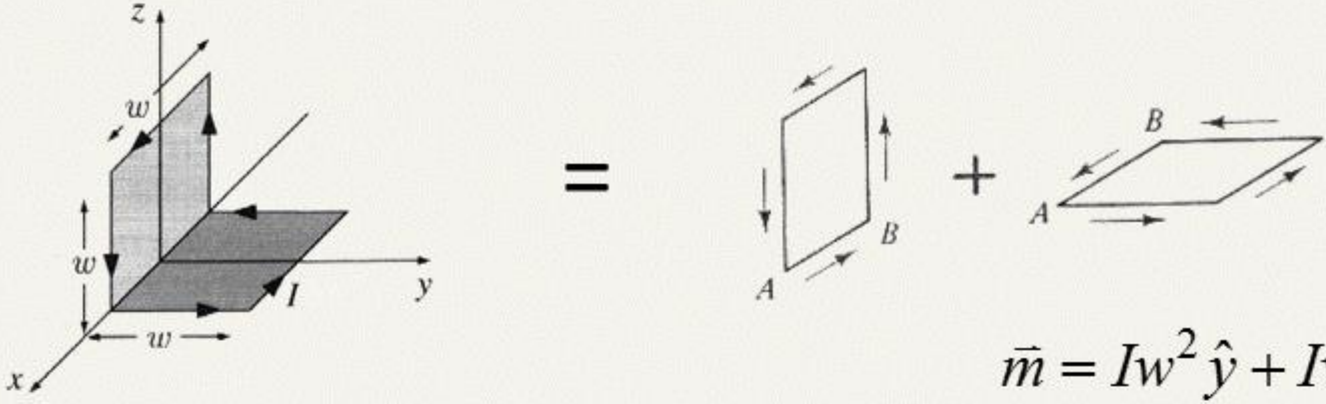
$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2}\left[-\frac{1}{2}\hat{r}\times\oint(\vec{r}'\times d\vec{r}')\right]$$

$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

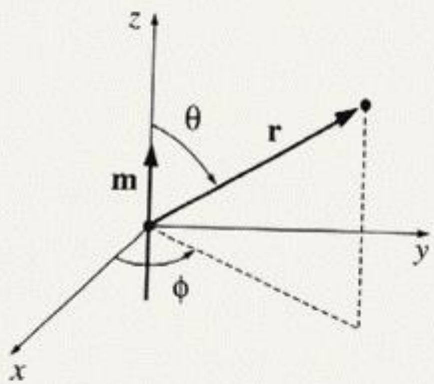
$\vec{m} = \frac{I}{2}\oint(\vec{r}'\times d\vec{l})$ **magnetic dipole moment**

$\vec{m} = I\vec{a}$ $\quad \vec{a} = \frac{1}{2}\oint(\vec{r}'\times d\vec{l})$ vector area

Example: Magnetic dipole moment of a bookend shaped loop:



$$\vec{m} = Iw^2\,\hat{y} + Iw^2\,\hat{z}$$



$$\vec{A}_{dip}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$

$$= \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

$$\vec{B} = \nabla\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi)\right]\hat{r} + \frac{1}{r}\left[-\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta}$$

$$= \frac{\mu_0}{4\pi}\frac{m}{r^3}(2\cos\theta\hat{r} + \sin\theta\hat{\theta})$$