3. Calculating Electrostatic Potential

3.1 Laplace's Equation

3.2 The Method of Images

3.3 Separation of Variables

3.4 Multipole Expansion

3.1.1 Introduction

The primary task of electrostatics is to study the interaction (force) of a given stationary charges.

since $\vec{F} = q_{test}\vec{E}$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int(\frac{\hat{R}}{R^2})\rho\, d\tau$$

$\because$ this integrals can be difficult (unless there is symmetry)

$$\vec{E} = E_x\hat{x} + E_y\hat{y} + E_z\hat{z} + \cdots$$

$\therefore$ we usually calculate

$$V = \frac{1}{4\pi\varepsilon_0}\int(\frac{1}{R})\rho\, d\tau$$

$$\vec{E} = -\nabla V$$

This integral is often too tough to handle analytically.

3.1.1

In differential form $\nabla^2 V = -\dfrac{\rho}{\varepsilon_0}$ (Poisson's equation)

- to solve a differential eq. we need boundary conditions.
- In case of $\rho = 0$, Poisson's eq. reduces to Laplace's equation

$$\nabla^2 V = 0$$

Or in general

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

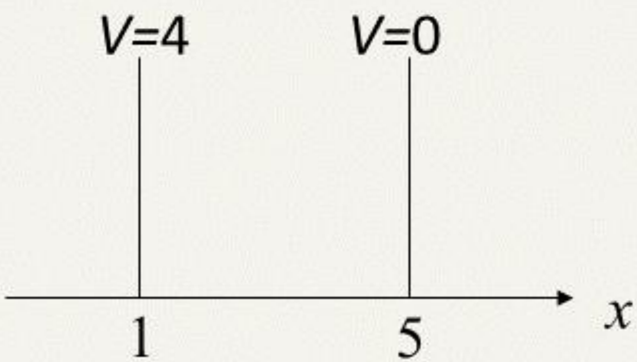
The solutions of Laplace's equation are called harmonic function.

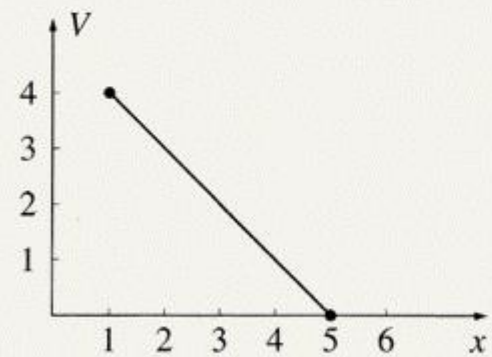
3.1.2 Laplace's Equation in One Dimension

$$\frac{d^2V}{dx^2} = 0 \;\Rightarrow\; V = mx + b$$

m*, *b are to be determined by B.C.s

e.q., $\begin{cases} V(x=1) = 4 \\ V(x=5) = 0 \end{cases} \Rightarrow \begin{cases} m = -1 \\ b = 5 \end{cases} \Rightarrow V = -x + 5$





1. ***V***(*x*) is the average of ***V***(*x* + *R*) and ***V***(*x* - *R*), for any *R*:

$$V(x) = \tfrac{1}{2}\left[V(x+R) + V(x-R)\right]$$

⇒ Laplace's equation tolerates no local maxima or minima.

3.1.3 Laplace's Equation in Two Dimensions

A partial differential eq. :

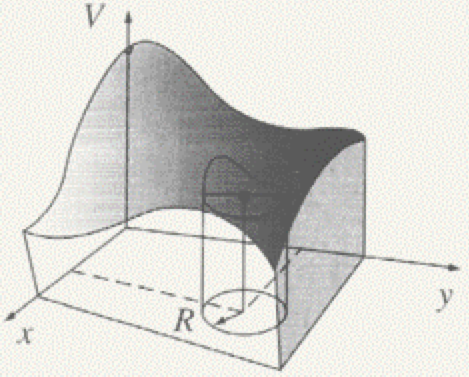
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

There is no general solution.
We discuss certain general properties for now.

1. The value of ***V*** at a point (*x*, *y*) is the average of those around the point.

$$V(x, y) = \frac{1}{2\pi R} \oint_{circle} V dl$$

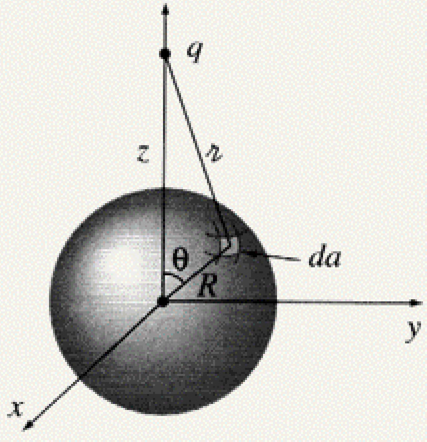
2. ***V*** has no local maxima or minima; all extreme occur at the boundaries.



3.1.4 Laplace's Equation in Three Dimensions

The value of V at point P is the average value of V over a spherical surface of radius R centered at P:

$$V(p) = \frac{1}{4\pi R^2} \oint_{sphere} V da$$



As a consequence, V can have no local maxima or minima, the extreme values of V must occur at the boundaries.

Example:

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{\mathscr{r}}, \ \mathscr{r}^2 = r^2 + R^2 - 2rR\cos\theta$$

$$V_{ave} = \frac{1}{4\pi R^2}\frac{q}{4\pi\varepsilon_0}\int [r^2 + R^2 - 2rR\cos\theta]^{-\frac{1}{2}} R^2 \sin\theta d\theta d\varphi$$

$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{2rR}\sqrt{r^2 + R^2 - 2rR\cos\theta}\Big|_0^\pi$$

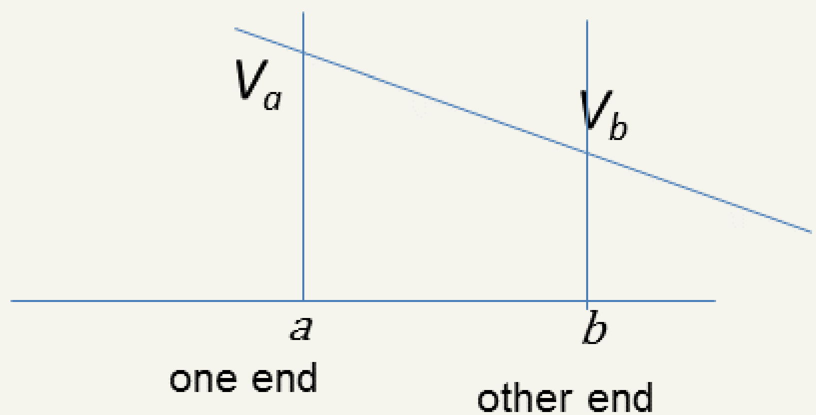
$$= \frac{q}{4\pi\varepsilon_0}\frac{1}{2rR}\left[(r+R)-(r-R)\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} = V_{\text{at the center of the sphere}}$$

The same for a collection of q by the superposition principle.

3.1.5 Boundary Conditions and Uniqueness Theorems

First uniqueness theorem :

The solution to Laplace's equation in some region is uniquely determined, if the value of ***V*** is specified on all their surfaces;
The outer boundary could be at infinity, where ***V*** is ordinarily taken to be zero.



in 1D,

$$V = mx + b$$

V is uniquely determined by its value at the boundary.

Proof: Suppose V_1, V_2 are two solutions for the same boundary conditions.

$$\nabla^2 V_1 = 0 \qquad \nabla^2 V_2 = 0$$

$$V_3 = V_1 - V_2$$

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

at boundary $V_1 = V_2 \quad \Rightarrow \quad V_3 = 0$ at boundary.

$\therefore \quad V_3 = 0$ everywhere

hence $V_1 = V_2$ everywhere

3.1.5

The first uniqueness theorem also applies to regions with charge.

Proof. $\nabla^2 V_1 = -\frac{\rho}{\varepsilon_0}$ $\quad\quad$ $\nabla^2 V_2 = -\frac{\rho}{\varepsilon_0}$

$$V_3 = V_1 - V_2$$

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = -\frac{\rho}{\varepsilon_0} + \frac{\rho}{\varepsilon_0} = 0$$

at boundary.

$$V_3 = V_1 - V_2 = 0$$

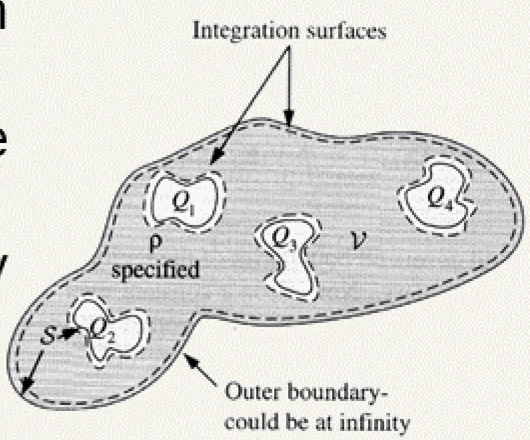
$$\therefore V_3 = 0, \quad i.e., \; V_1 = V_2$$

Corollary : The potential in some region is uniquely determined if
(a) the charge density throughout the region, and
(b) the value of V on all boundaries, are specified.

3.1.6 Conductors and the Second Uniqueness Theorem

Second uniqueness theorem:

In a region containing conductors and filled with a specified charge density ρ,
the electric field is uniquely determined if the total charge on each conductor is given.
(The region as a whole can be bounded by another conductor, or else unbounded.)



Proof:

Suppose both $\vec{E}_1$ and $\vec{E}_2$ satisfy the same configuration .

$$\nabla \cdot \vec{E}_1 = \frac{1}{\varepsilon_0}\rho \quad \nabla \cdot \vec{E}_2 = \frac{1}{\varepsilon_0}\rho$$

and

$$\oint_{\substack{ith\ conducting\\ surface}} \vec{E}_1 \cdot d\vec{a}_1 = \frac{1}{\varepsilon_0}Q_i, \quad \oint_{\substack{ith\ conducting\\ surface}} \vec{E}_2 \cdot d\vec{a}_2 = \frac{1}{\varepsilon_0}Q_i$$

$$\oint_{outer\ bourndary} \vec{E}_1 \cdot d\vec{a} = \frac{1}{\varepsilon_0}Q_{tot}, \quad \oint_{outer\ bourndary} \vec{E}_2 \cdot d\vec{a} = \frac{1}{\varepsilon_0}Q_{tot}$$

define $\vec{E}_3 = \vec{E}_1 - \vec{E}_2$

$$\nabla \cdot \vec{E}_3 = \nabla \cdot (\vec{E}_1 - \vec{E}_2) = \nabla \cdot \vec{E}_1 - \nabla \cdot \vec{E}_2 = \frac{\rho}{\varepsilon_0} - \frac{\rho}{\varepsilon_0} = 0$$

for region in between the conductors, and

$$\oint \vec{E}_3 \cdot d\vec{a} = \oint \vec{E}_1 \cdot d\vec{a} - \oint \vec{E}_2 \cdot d\vec{a} = 0 \text{ over each boundary}$$

$\boldsymbol{V}_3$ is a constant over each conducting surface, using

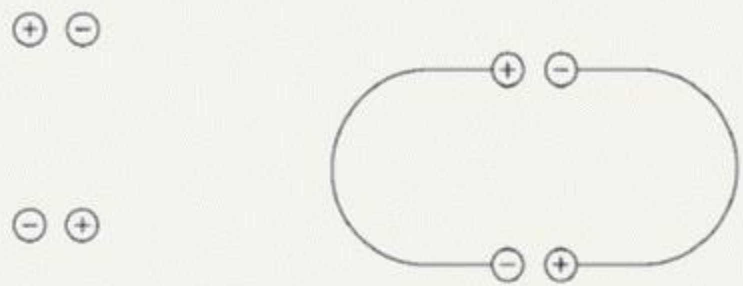
$$\nabla \cdot (V_3 \vec{E}_3) = V_3 (\nabla \cdot \vec{E}_3) + \vec{E}_3 \cdot (\nabla V_3) = -E_3^2$$

$$\int_{volume} E_3^2 d\tau = -\int \nabla \cdot (V_3 \vec{E}_3) d\tau = -\oint_{surface} V_3 \vec{E}_3 \cdot d\vec{a}$$

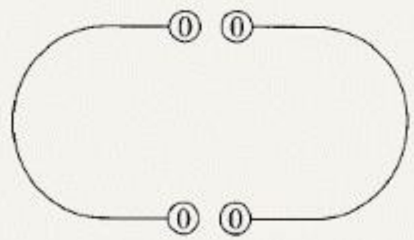
$$= -V_3 \oint_{surface} \vec{E}_3 \cdot d\vec{a} = 0$$

$\therefore \quad \vec{E}_3 = 0$ i.e., $\vec{E}_1 = \vec{E}_2$

Example (by Purcell's):



Is this charge configuration possible? Total charge in each conductor is zero



Same total charge as above, so this must be the stable configuration solution.

3.2 The Method of Images

3.2.1 The Classical Image Problem

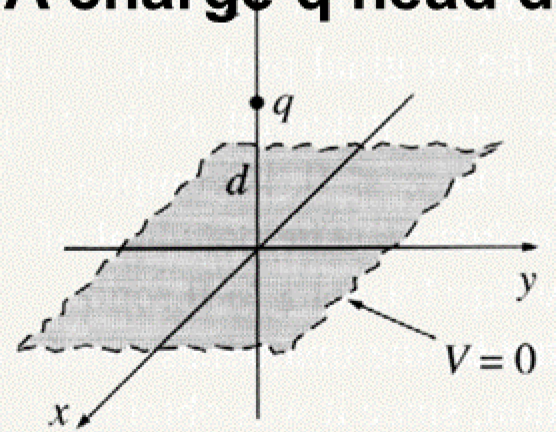
3.2.2 The Induced Surface Charge

3.2.3 Force and Energy

3.2.4 Other Image Problems

3.2 The Method of Images

A charge q head d above an infinite grounded plane:



Boundary conditions:

1. $V = 0$ when $z = 0$, since the plane is grounded
2. $V \to 0$ far from the charge,

ie. $x^2 + y^2 + z^2 >> d^2$

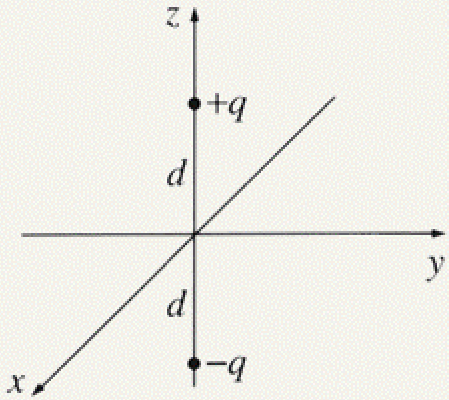
What is $\boldsymbol{V}$ (z>0) ?

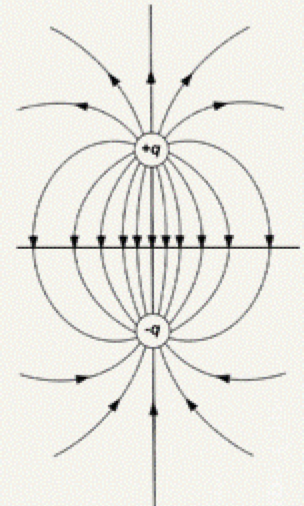
The first uniqueness theorem guarantees that there is only one solution.

If we can get one by any means, that is the only answer.

3.2.1

Trick :





Forget the plane, consider another charge –q at (0,0,-d), for this configuration.

$$V(x,y,z)=\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}}+\frac{-q}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

$$\mathrm{V}=0 \text{ at } \mathrm{z}=0 \text{ and } V\to 0 \text{ for } x^2+y^2+z^2>>d^2$$

Has same boundary conditions as the original problem, so by the uniqueness theorem this is the solution for original problem for z>0

$\vec{E}(z<0)=0$ in the original problem but we only care $z>0$, $z<0$ is not a concern

3.2.2 The Induced Surface Charge

$$E_z\Big|_{z=0} = \frac{\sigma}{\varepsilon_0} \Rightarrow \sigma = -\varepsilon_0 \frac{\partial V}{\partial z}\Bigg|_{z=0}$$

$$\sigma = \frac{-1}{4\pi}\frac{\partial}{\partial z}\left\{\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right\}\Bigg|_{z=0}$$

$$\sigma(x,y) = \frac{-qd}{2\pi\left(x^2+y^2+(z-d)^2\right)^{3/2}}$$

$$\sigma(r,z=0) = \frac{-qd}{2\pi\left(r^2 \;\; +d^2\right)^{3/2}}$$

$$\text{total induced charge } Q = \int_0^{2\pi}\int_0^{\infty}\frac{-qd}{2\pi\left(r^2 \;\; +d^2\right)^{3/2}} r\,dr\,d\phi = -q$$

3.2.3 Force and Energy

The charge q is attracted toward the plane.
The force of attraction is

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q(-q)}{\left[d-(-d)\right]^2}\hat{z} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{z}$$

With 2 point charges and no conducting plane, the energy is

$$W = \frac{1}{2}\sum_{i=1}^{2} q_i V_i(p_i) \qquad \text{Eq.(2.36)}$$

$$= \frac{1}{2}\left\{(q)\cdot\frac{1}{4\pi\varepsilon_0}\left[-\frac{q}{d+d}\right] + (-q)\cdot\frac{1}{4\pi\varepsilon_0}\left[\frac{q}{\sqrt{(-d-d)^2}}\right]\right\}$$

$$= -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)}$$

3.2.3 (2)

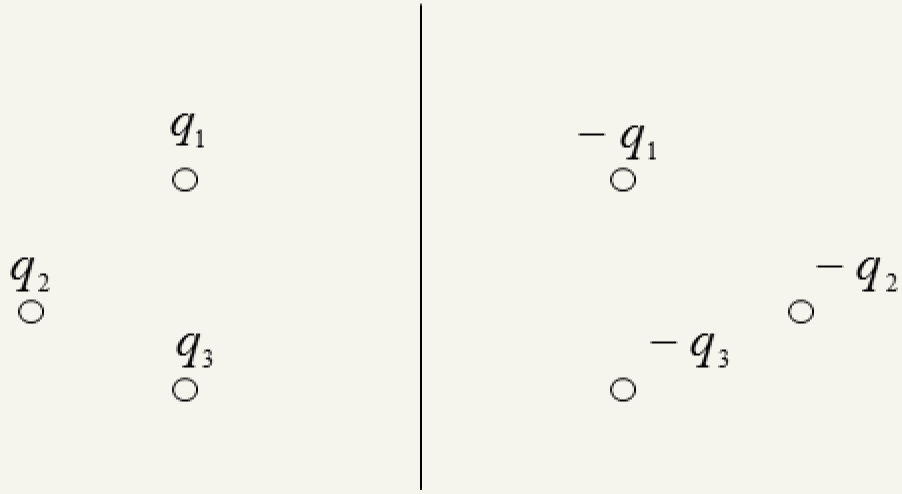
For point charge q and the conducting plane at $z = 0$ the energy is half of the energy given at above, because the field exist only at $z \geq 0$,and is zero at $z < 0$; that is

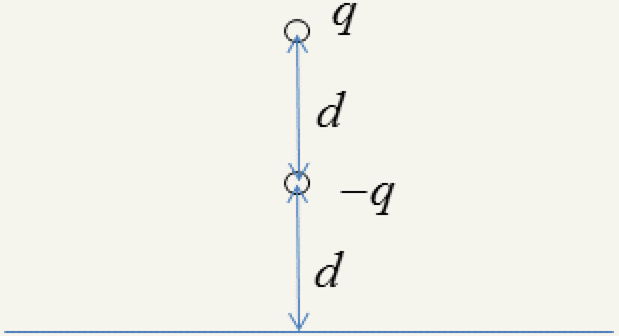
$$W = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(4d)} \quad \text{or}$$

$$W = \int_{\infty}^{d} \vec{F}\cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0}\int_{\infty}^{d}\frac{q^2}{(2z)^2}dz \qquad (\because d\vec{l} = -dl\hat{z})$$

$$= \frac{1}{4\pi\varepsilon_0}\left(-\frac{q^2}{4z}\right)\bigg|_{\infty}^{d} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(4d)}$$

In general for any stationary distribution of charge

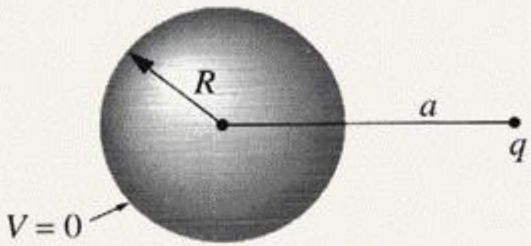




Find the force on charge q.

3.2.4 (2)

Conducting sphere of radius *R at V=0*



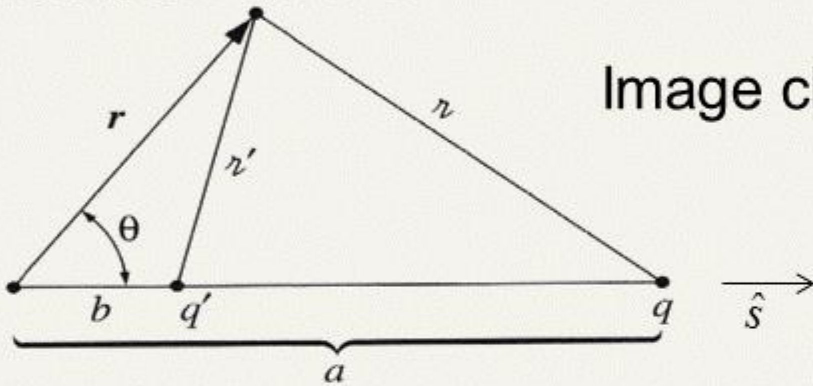


Image charge $q' = -\frac{R}{a}q$

at $b = \frac{R^2}{a}$

$$\mathscr{r} = \left[r^2 + a^2 - 2ra\cos\theta\right]^{\frac{1}{2}}$$

$$\mathscr{r}' = \left[r^2 + b^2 - 2rb\cos\theta\right]^{\frac{1}{2}}$$

$$V(r,\theta) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right)$$

$$= 0 \text{ when } r = R$$

Force, $$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{(a-b)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}$$

3.3 Separation of Variables

3.3.0 Fourier series and Fourier transform

3.3.1 Cartesian Coordinate

3.3.2 Spherical Coordinate

3.3 Completeness and Orthogonality:

Basic set of unit vectors in a certain coordinate can express any vector uniquely in the space represented by the coordinate.

e.g. $\vec{V} = \sum_{i=1}^{N} V_i \hat{i} = V_x \hat{x} + V_y \hat{y} + V_z \hat{z}$ in 3D. Cartesian Coordinates.

V_x, V_y, V_z are unique because $\hat{x}, \hat{y}, \hat{z}$ are orthogonal. $\hat{i} \cdot \hat{j} = 0 \quad i \neq j$

$= 1 \quad i = j$

These ideas can be extended to functions, for example functions defined in an interval (a,b) can be considered as a vector space of functions.

Completeness: a set of functions $f_n(x)$ is complete if for any function $f(x)$

$$f(x) = \sum_{n=1}^{\infty} C_n f_n(x)$$

Orthogonal: a set of functions is orthogonal if

$$\int_a^b f_n(x) f_m(x) dx = 0 \qquad \text{for} \quad n \neq m$$

$$= \text{const} \qquad \text{for} \quad n = m$$

A complete and orthogonal set of functions forms a basic set of functions. e.g.

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0 & if \quad m \neq n \\ \pi & if \quad m = n \end{cases} \qquad k, n \in N$$

$\sin(nx)$ is an orthoganal set of functions in the $[-\pi, \pi]$ range

Since they are odd: $\sin\left(n(-x)\right) = -\sin\left(nx\right)$ sinnx is a basic set of fuctions for any odd function in $[-\pi, \pi]$

Similarly cos(nx) is a set of basic orthonomal functions for any even function in $[-\pi, \pi]$

$$\cos\left(n(-x)\right) = \cos\left(nx\right) \; ; \; \int_{-\pi}^{\pi} \cos(mx)\cos(nx) \, dx = \begin{cases} 0 & if \quad m \neq n \\ \pi & if \quad m = n \end{cases}$$

Since $\int_{-\pi}^{\pi} \sin(mx)\cos(nx) \, dx = \begin{cases} 0 & if \quad m \neq n \\ \pi & if \quad m = n \end{cases}$

sin(nx) and cos(nx) is a set of basic orthonomal functions for any function in $[-\pi, \pi]$

for any function $f(x)$

$$g(x)=\frac{f(x)-f(-x)}{2} \text{ is odd;} \quad h(x)=\frac{f(x)+f(-x)}{2} \text{ is even}$$

$$f(x)=\underset{\text{odd}}{\underset{\uparrow}{g(x)}}+\underset{\text{even}}{\underset{\uparrow}{h(x)}}$$

Fourier series is expressing a function in terms of basic functions $\sin nx$ and $\cos nx$

$$f(x)=\sum_{n=0}^{\infty}\left(A_n \sin nx + B_n \cos nx\right)$$

$$\left.\begin{aligned} A_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \\ B_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \end{aligned}\right\} \quad n=1,2,\cdots\infty$$

$$B_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad A_0 = 0$$

Proof:

$$\int_{-\pi}^{\pi} f(x)\cdot\sin kx dx = \sum_{n=0}^{\infty}\int_{-\pi}^{\pi}\left(A_n \sin nx + B_n \cos nx\right)\sin(kx)\,dx$$

$$= \sum_{n=0}^{\infty} A_n \int_{-\pi}^{\pi} \sin(nx)\sin(kx)\,dx$$

$$= A_k \pi \qquad \text{if} \quad k \neq 0$$

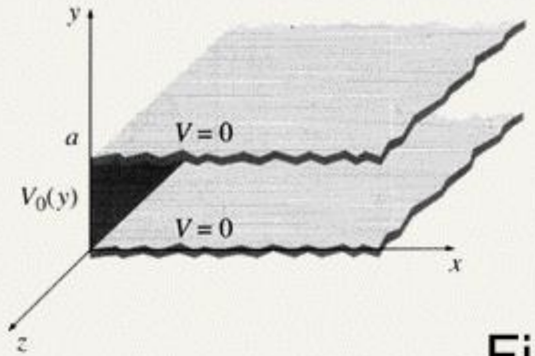
$$\Rightarrow \quad A_k = \frac{1}{\pi}\int_{0}^{\pi} f(x)\sin(kx)\,dx$$

Similarly:

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx$$

$$B_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

Example : Use the method of separation of variables to solve the Laplace's eq.



$$V(y=0)=0$$

$$V(y=\pi)=0$$

$$V(x=0)=V_0(y)$$

$$V(x\to\infty)\to 0$$

Find the potential inside this "slot".

Laplace equation: $\dfrac{\partial^2 V}{\partial x^2}+\dfrac{\partial^2 V}{\partial y^2}=0$

$$V(x,y)=X(x)Y(y)$$

$$Y\frac{\partial^2 X}{\partial x^2}+X\frac{\partial^2 Y}{\partial y^2}=0 \quad \Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2}+\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=0 \quad \Rightarrow \quad \frac{1}{X}\frac{\partial^2 X}{\partial x^2}=-\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}=k^2$$

$$\frac{\partial^2 X}{\partial x^2}-k^2X=0 \quad \Rightarrow X=Ae^{kx}+Be^{-kx}$$

$$\frac{\partial^2 Y}{\partial y^2}+k^2Y=0 \Rightarrow X=C\sin kx+D\cos kx$$

B.C. (iv) $V(x \to \infty) \to 0 \quad \Rightarrow \quad A = 0, \quad k > 0$

$$V(x, y) = e^{-kx}(C \sin ky + D \cos ky)$$

B.C. (i) $V(y = 0) = 0 \quad \Rightarrow \quad D = 0$

$$V(x, y) = Ce^{-kx} \sin ky$$

B.C. (ii) $V(y = a) = 0 \quad \Rightarrow \quad \sin ka = 0 \Rightarrow k_n = \dfrac{n\pi}{a} \quad n = 1, 2, 3 \cdots$

According to the principle of superposition $V(x, y) = \sum_{n=1}^{\infty} Ce^{-\frac{n\pi x}{a}} \sin \dfrac{n\pi y}{a}$

B.C. (iii) $V(x = 0) = V_0(y) \quad \Rightarrow V_0(y) = \sum_{n=1}^{\infty} C_n e^{-k_n x} \sin k_n y$

A Fourier series for odd function

$$C_n = \frac{2}{a} \int_0^a V_0(y) \sin k_n y \, dy$$

$$\left(\because \int_0^a \sin \frac{n\pi y}{a} \sin \frac{m\pi y}{a} dy = 0 \ \mathrm{n} \neq \mathrm{m} \atop = \frac{a}{2} \right)$$

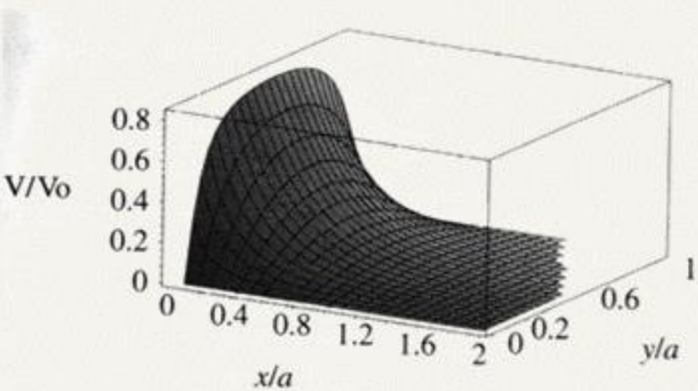
3.3.1 (4)

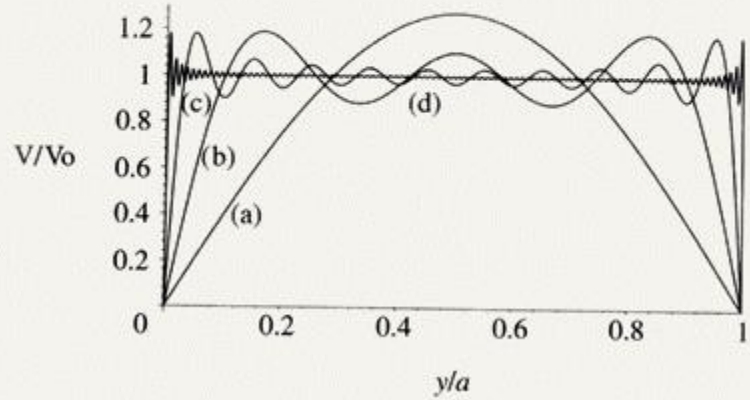
For $V_0(y) = V_0 = constant$

$$C_n = \frac{2V_0}{a}\int_0^a \sin\frac{n\pi y}{a}\,dy$$

$$= \frac{2V_0}{n\pi}(1-\cos n\pi) = \begin{cases} 0 & if\ \ n = even \\ \frac{4V_0}{n\pi} & if\ \ n = odd \end{cases}$$

$$V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\cdots}\frac{1}{k_n}e^{-k_n}\sin k_n y = \left(\frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin^{n\pi y}/_a}{\sinh^{\pi x}/_a}\right)\right)$$





Laplace equation in spherical coordinates :

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial^2\phi^2}=0$$

In cases of azimuthal symmetry V is independent of ϕ so $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$

Look for a solution $V(r,\theta)=R(r)\Theta(\theta)$ $\Rightarrow\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)+\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0$

$$\Rightarrow\quad\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)=\text{constant}=l(l+1)\quad\text{and}\quad\frac{1}{\Theta\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=-l(l+1)$$

$\Rightarrow\quad\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=l(l+1)$ try a solution $R=r^n\Rightarrow n(n+1)=l(l+1)$

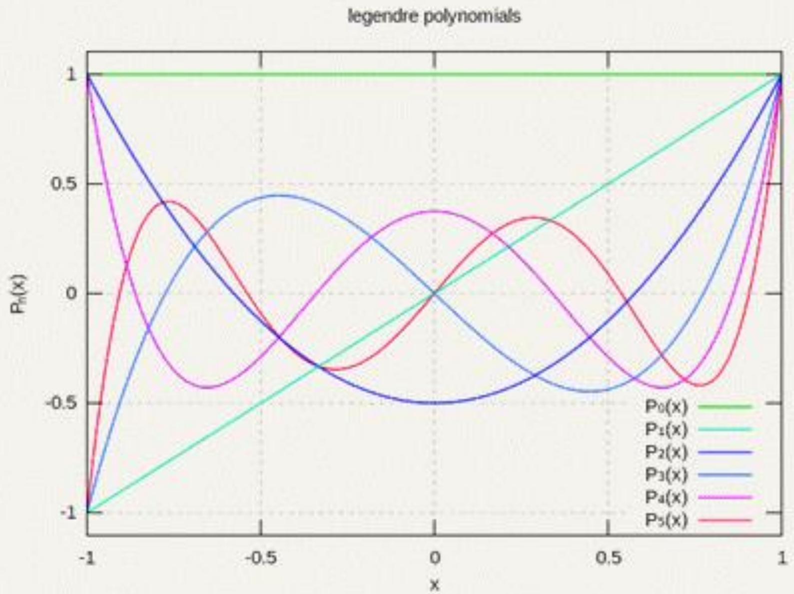
$n=l\ or\ n=-(l+1)\Rightarrow R(r)=Ar^l+\frac{B}{r^{l+1}}$

and $\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=-l(l+1)\Theta\sin\theta$

subsitute $\cos\theta=x$ $\Rightarrow(1-x^2)\frac{d^2\Theta}{dx^2}-2x\frac{d\Theta}{dx}+l(l+1)\Theta=0$

solutions of $\left(1-x^2\right)\dfrac{d^2\Theta}{dx^2}-2x\dfrac{d\Theta}{dx}+l(l+1)\Theta=0$

are called the *Legendre* polynomials $P_l(x)$



$$P_0(x)=1$$
$$P_1(x)=x$$
$$P_2(x)=\left(3x^2-1\right)/2$$
$$P_3(x)=\left(5x^3-3x\right)/2$$
$$P_4(x)=\left(35x^4-30x^2+3\right)/8$$
$$P_5(x)=\left(63x^5-70x^3+15x\right)/8$$

In general given by the **Rodrigues formula** $P_l(x)=\dfrac{1}{2^l l!}\left(\dfrac{d}{dx}\right)^l\left(x^2-1\right)^l$

so the soltion for $\Theta(\theta)=P_l(\cos\theta)$

and the seperable solution for $V(r,\theta)$ is

$$V(r,\theta)=\left(Ar^l+\frac{B}{r^{l+1}}\right)P_l(\cos\theta)$$

The genetal solution is $$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

- Legendre polynomials $P_n(x)$ are a complete, orthogonal set of functions in the interval $-1<x<1$ they satisfy

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \int_{0}^{\pi} P_n(\cos\theta)P_m(\cos\theta)\sin\theta d\theta = 0 \text{ if n} \neq \text{m}$$

$$= \frac{2}{2l+1} \quad \text{if n=m}$$

Example: The potential $V_0(\theta$ is specified on the surface of a hollow sphere of radius R, find the potential inside the sphere.

In this case $B_l = 0$ for all l, since potential has to be finite at the origin(center)

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

at $r = R$ $\quad V(r,\theta) = V_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos\theta)$

Using orthoganality relations $\quad \text{A}_l R^l \dfrac{2}{2l+1} = \int_{0}^{\pi} V_0 P_{l'}(\cos\theta)\sin\theta d\theta$

$$A_l = \frac{(2l+1)}{2R^l}\int_0^{\pi} V_0 P_l(\cos\theta)\sin\theta d\theta$$

$$V(r,\theta) = \sum_{l=0}^{\infty} AR^l P_l(\cos\theta)$$ where A is given by above formula

To find the potential outside the sphere:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

Now A_l must be zero ($V\to 0$ as $r\to\infty$)

$$V(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)$$

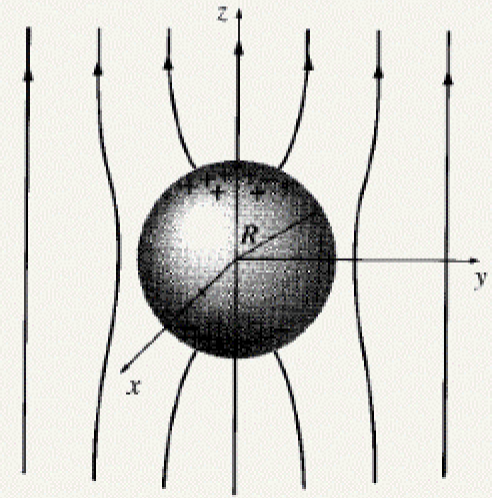
on the surface of the sphere: $V(R,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta) = V_0(R,\theta)$

$$\frac{B_l}{R^{l+1}}\frac{2}{2l+1} = \int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta d\theta$$

$$B_l = \frac{(2l+1)}{2}R^{l+1}\int_0^{\pi} V_0(\theta)P_l(\cos\theta)\sin\theta d\theta$$

Example:

An uncharged conductive sphere of Radius R is placed in an electric field $\vec{E} = E\hat{z}$ What is the resulting field distribution due to induced charges on the sphere.



$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos\theta)$$

$V = E_0 z + C$; choose V=0 at z=0 $\Rightarrow$ C=0

$V = E_0 r\cos\theta$ for r>>R

$V = 0$ when r=R $\Rightarrow A_l R^l + \dfrac{B_l}{R^{l+1}} = 0 \Rightarrow B_l = A_l R^{2l+1}$

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l\left(r^l + \frac{R^{2l+1}}{r^{l+1}}\right)P_l(\cos\theta)$$

for r>>R second term is neglegible $\displaystyle\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r\cos\theta$

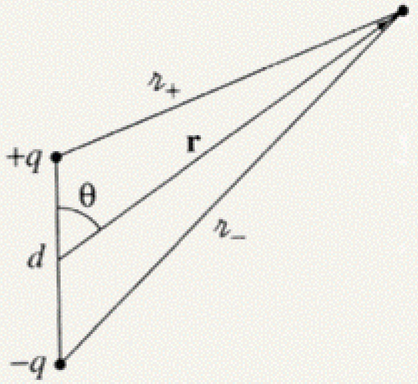
$\Rightarrow A_1 = -E_0$ all other $A_l = 0$ $\qquad V(r,\theta) = -E_0\left(r - \dfrac{R^3}{r^2}\right)\cos\theta$

3.4 Multipole Expansion:

Approximate Potentials at Large distances

- An **electric dipole** consists of two charges $+q$ and $-q$ separated by distance d. It is neutral but produces an $\vec{E}$-field at points far from the dipole.

$$V(p)=\frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\mathscr{r}_+}-\frac{q}{\mathscr{r}_-}\right)$$



$$\mathscr{r}_+^{\,2}=r^2+\left(\frac{d}{2}\right)^2-rd\cos\theta;\ \mathscr{r}_-^{\,2}=r^2+\left(\frac{d}{2}\right)^2+rd\cos\theta$$

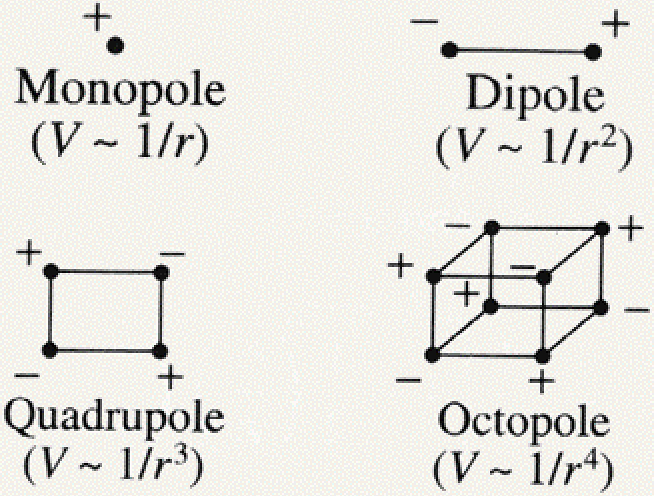
$$\mathscr{r}_\pm^{\,2}=r^2\left(1\mp\frac{d}{r}\cos\theta+\frac{d^2}{4r^2}\right)$$

$$\overset{r>>d}{\cong}\ r^2\left(1\mp\frac{d}{r}\cos\theta\right)$$

$$\frac{1}{\mathscr{r}_\pm}\cong\frac{1}{r}\left(1\mp\frac{d}{r}\cos\theta\right)^{-1/2}\cong\frac{1}{r}\left(1\pm\frac{d}{2r}\cos\theta\right)$$

$$\left(\frac{1}{\mathscr{r}_+}-\frac{1}{\mathscr{r}_-}\right)\cong\frac{d}{r^2}\cos\theta$$

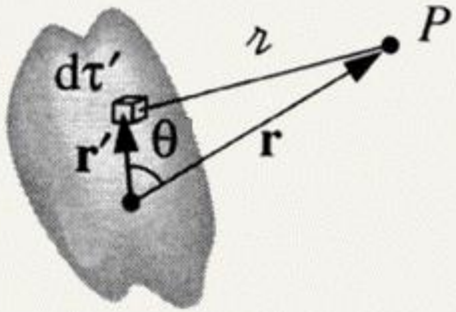
$$\Rightarrow\quad V(p)\cong\frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}$$



3.4.1

For an arbitrary localized charge distribution.

$$V(p) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{\mathscr{r}}\rho(\vec{r}\,')\,d\tau'$$



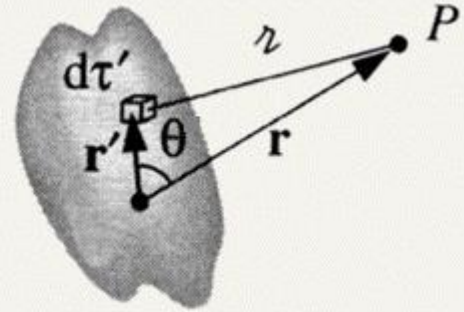
$$\mathscr{r}^2 = r^2 + r'^2 - 2rr'\cos\theta$$

$$= r^2\left[1 + \left(\frac{r'}{r}\right)^2 - 2\left(\frac{r'}{r}\right)\cos\theta\right]$$

$$\mathscr{r} = r\sqrt{1+\varepsilon} \qquad \varepsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta\right)$$

for $r >> r'$ $\epsilon << 1$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}(1+\varepsilon)^{-1/2} = \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \frac{5}{16}\varepsilon^3 + \cdots\right)$$



$$\frac{1}{\mathscr{r}} = \frac{1}{r}[1 - \frac{1}{2}(\frac{r'}{r})(\frac{r'}{r} - 2\cos\theta) + \frac{3}{8}\left(\frac{r'}{r}\right)^2\left(\frac{r'}{r} - 2\cos\theta\right)^2 - \frac{5}{16}\left(\frac{r'}{r}\right)^3\left(\frac{r'}{r} - 2\cos\theta\right)^3 + \cdots]$$

$$= \frac{1}{r}[1 + \left(\frac{r'}{r}\right)\cos\theta + \left(\frac{r'}{r}\right)^2\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right) + \left(\frac{r'}{r}\right)^3\left(\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta\right) + \cdots]$$

$$= \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta)\rho\, d\tau' = \sum_{n=0}^{\infty}\frac{r'^n}{r^{n+1}}P_n(\cos\theta)\rho\, d\tau'$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{(n+1)}}\int(r')^n P_n(\cos\theta)\rho(r')\, d\tau'$$

$$= \frac{1}{4\pi\varepsilon_0}[\frac{1}{r}\int\rho(r')\, d\tau' + \frac{1}{r^2}\int r'\cos\theta\,\rho(r')\, d\tau' + \frac{1}{r^3}\int(r')^2(\frac{3}{2}\cos^2\theta - \frac{1}{2})\rho(r')\, d\tau' + \cdots]$$

Monopole term — Dipole term — Quadrupole term

Multipole expansion

3.4.2 The Monopole and Dipole Terms

monopole $$V_{mon}(p) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$ dominates if $r >> 1$

dipole $$V_{dip}(p) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\int r'\cos\theta\,\rho\,d\tau$$ $$r'\cos\theta = \hat{r}\cdot\vec{r}'$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{r}\cdot\underbrace{\int \vec{r}'\rho\,d\tau}_{\vec{P}}$$

dipole moment (vector)

$$V_{dip}(p) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

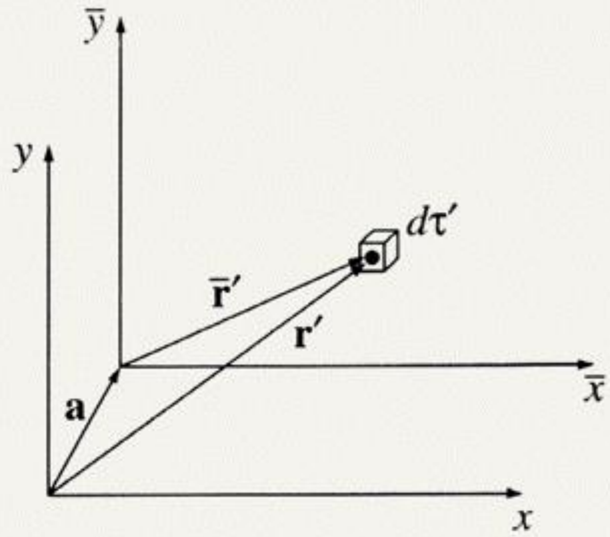
$$\vec{p} = \int \vec{r}'\rho\,d\tau \left(= \sum_{i=1}^{n} q_i\vec{r}_i' \text{ for point charges} \right)$$

A physical dipole is consist of a pair of equal and opposite charge, $\pm q$

$$\vec{p} = q\vec{r}_+' - q\vec{r}_-' = q(\vec{r}_+' - \vec{r}_-') = q\vec{d}$$

3.4.3 Origin of Coordinates in Multipole Expansions

Dependence of dipole moment on coordinate origin:



$$\bar{p} = \int \bar{r}' \rho \, d\tau$$

$$= \int (\vec{r}' - \vec{a}) \rho \, d\tau$$

$$= \int \vec{r}' \rho \, d\tau - \vec{a} \int \rho \, d\tau$$

$$= \vec{p} - \vec{a} Q$$

if $\quad Q = 0 \qquad \bar{p} = \vec{p}$

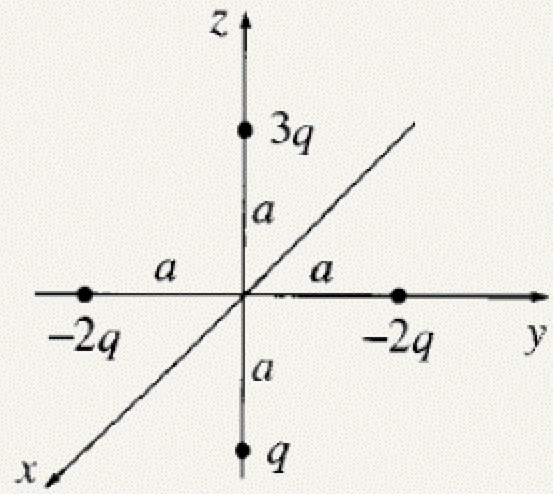
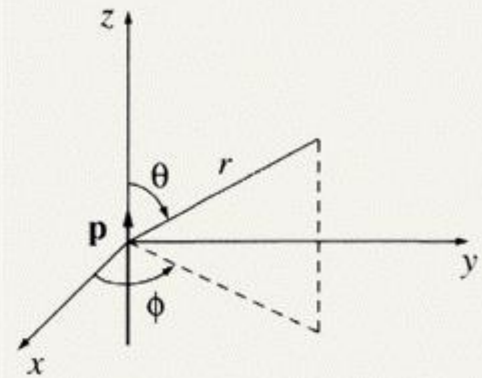


Figure 3.31

Problem 3.27 Four particles (one of charge q, one of charge $3q$, and two of charge $-2q$) are placed as shown in Fig. 3.31, each a distance a from the origin. Find a simple approximate formula for the potential, valid at points far from the origin. (Express your answer in spherical coordinates.)

3.4.4 The Electric Field of a Dipole

A pure dipole

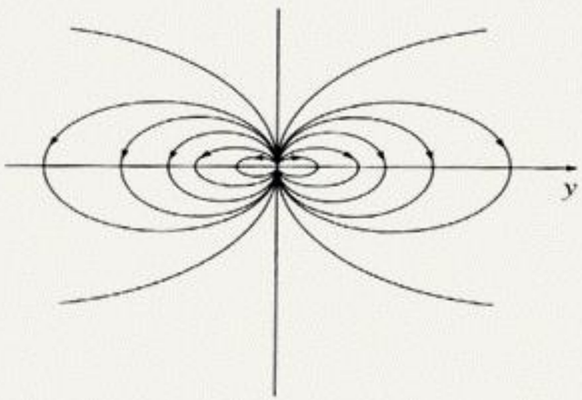


$$V_{dip}(r,\theta) = \frac{\vec{P}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{P\cos\theta}{4\pi\varepsilon_0 r^2}$$

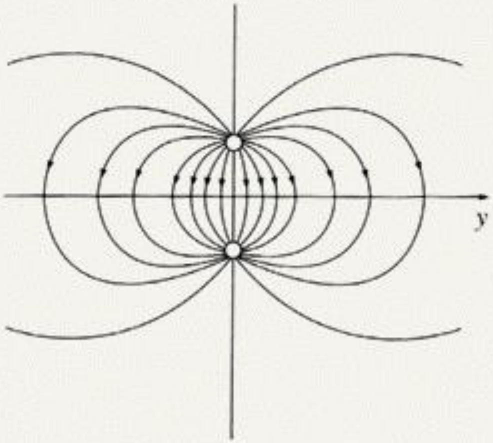
$$E_r = -\frac{\partial V}{\partial r} = \frac{2P\cos\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{P\sin\theta}{4\pi\varepsilon_0 r^3}$$

$$E_\varphi = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi} = 0$$



(a) Field of a "pure" dipole



(a) Field of a "physical" dipole

$$\vec{E}_{dip}(r,\theta) = \frac{P}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$