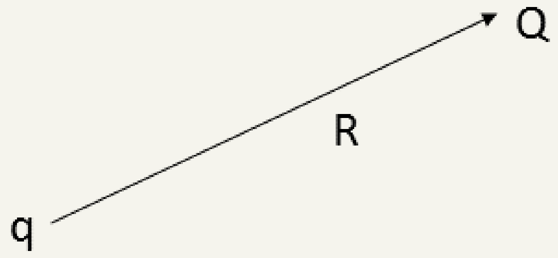
Electromagnetic Theory

PHYS 401

Electrostatics

- The Electrostatic Field
- Divergence and Curl of Electrostatic Field
- Electric Potential
- Work and Energy in Electrostatics
- Conductors

Coulomb's Law (2.1.2)

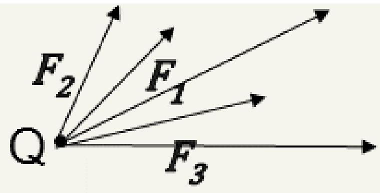


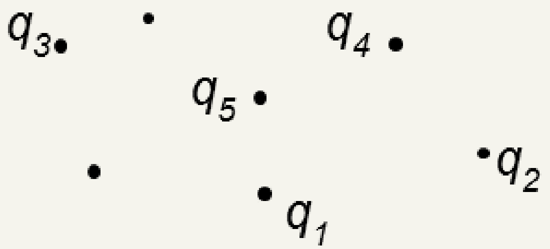
The force on a charge Q due to a single point charge q is given by Coulomb`s law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{R^2}\hat{R} \qquad \vec{R} = \vec{r}_Q - \vec{r}_q = R\,\hat{R}$$

$\varepsilon_0 = 8.85 \times 10^{-12} \dfrac{C^2}{N \cdot m^2}$ the permittivity of free space

Principle of Superposition





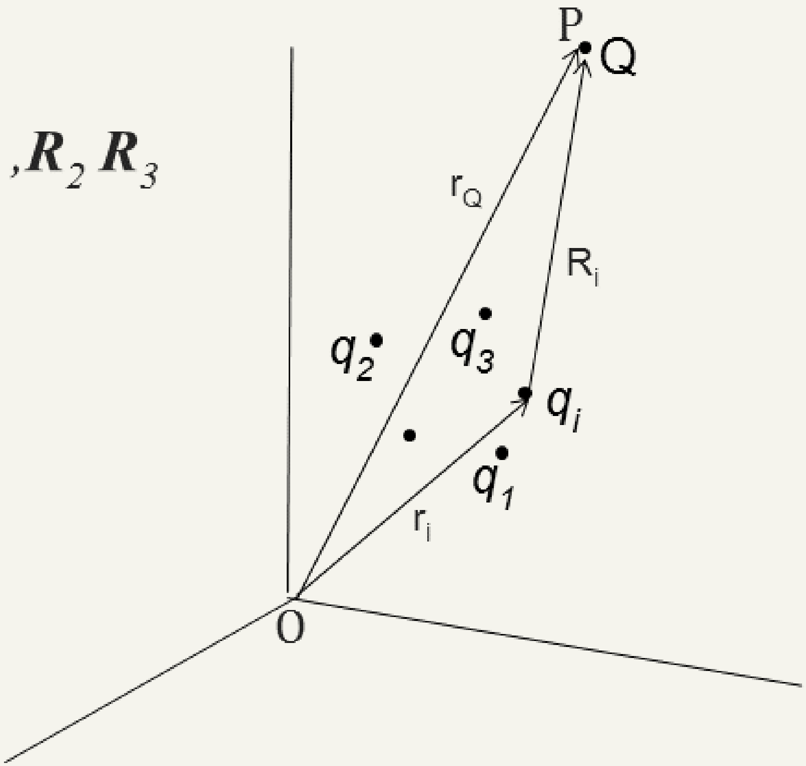
The interaction between any two charges is unaffected by the presence of other charges.
So the force on charge Q due to charges q_1 , q_2 q_3 … is the sum of forces due to q_1 , q_2 q_3

$$F = F_1 + F_2 + F_3 + \ldots$$

F_i is the force on Q due to q_i

The Electric Field (2.1.3)

If there are $q_1, q_2\ q_3$ charges at distances $\boldsymbol{R}_1, \boldsymbol{R}_2\ \boldsymbol{R}_3$ from the charge Q:



$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \cdots$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 Q}{R_1^2}\hat{R}_1 + \frac{q_2 Q}{R_2^2}\hat{R}_2 + \cdots\right)$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{q_1}{R_1^2}\hat{R}_1 + \frac{q_2}{R_2^2}\hat{R}_2 + \frac{q_3}{R_3^2}\hat{R}_3 + \cdots\right)$$

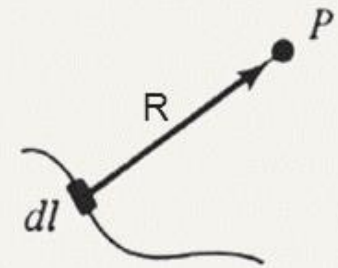
$$\vec{F} = Q\vec{E} \qquad \vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{R_i^2}\hat{R}_i$$

Where $\boldsymbol{R}_i = \boldsymbol{r}_Q - \boldsymbol{r}_i$

E(P) is the electric field at P due to source charges $q_1, q_2\ q_{3..}$ …

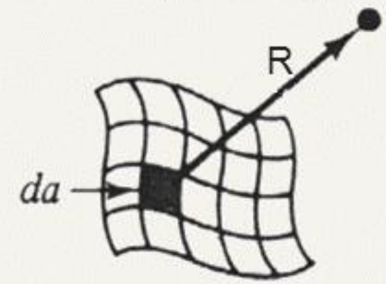
2.1.4 Continuous Charge Distributions

$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{line} \frac{\hat{R}}{R^2} \lambda \, d\ell$$



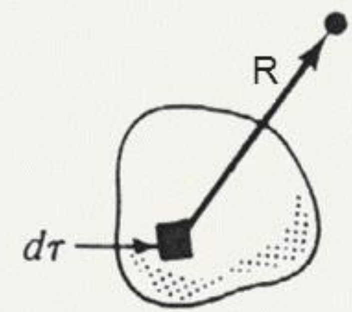
Linear charge density λ

$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{surface} \frac{\hat{R}}{R^2} \sigma \, da$$



Surface charge density σ

$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0} \int_{volume} \frac{\hat{R}}{R^2} \rho \, d\tau$$



Volume charge density ρ

Example: Find the electric field a distance z above the midpoint of a straight of length 2L, which carries a uniform line charge λ

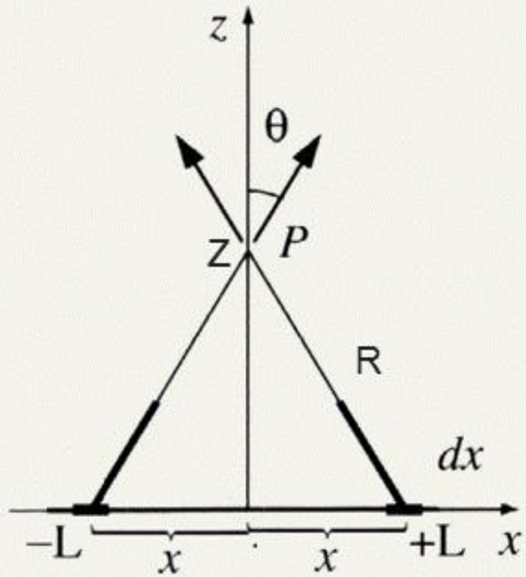
Solution:

$$dE = 2\frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda dx}{R^2}\right)\cos\theta\,\hat{z}$$

$$E = \frac{1}{4\pi\varepsilon_0}\int_0^L \frac{2\lambda z}{(z^2+x^2)^{3/2}}dx$$

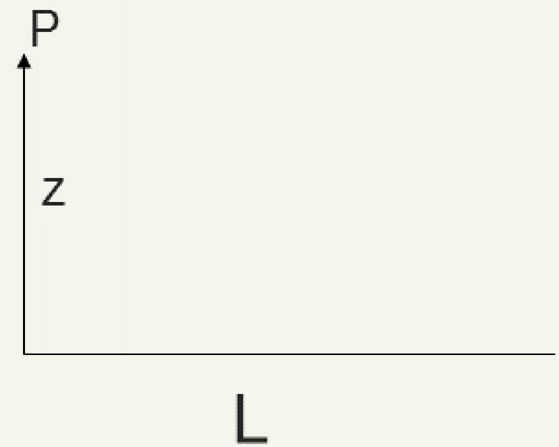
$$= \frac{2\lambda z}{4\pi\varepsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]\Bigg|_0^L$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z\sqrt{z^2+L^2}}$$



(1) z>>L $\vec{E} \cong \frac{1}{4\pi\varepsilon_0}\frac{2\lambda L}{z^2}$ (2) $L\rightarrow\infty$ $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{z}$

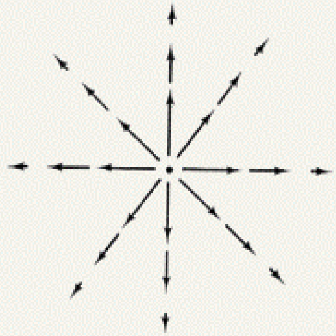
Problem: Find the electric filed at a distance z above one end of a straight line segment of length L with linear change density λ.



Divergence of Fields lines and the Gauss's law (2.2)

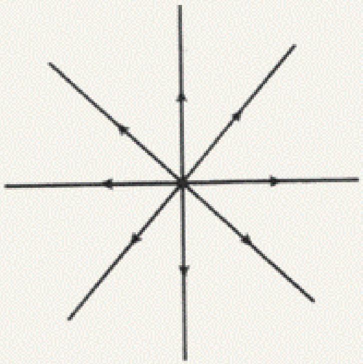
A single point charge q, situated at the origin

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$



Because the field falls off like $\frac{1}{r^2}$, electric field gets weaker as distance increases from the origin, and field always point radially outward, (shown as arrows with length proportional to field strength).

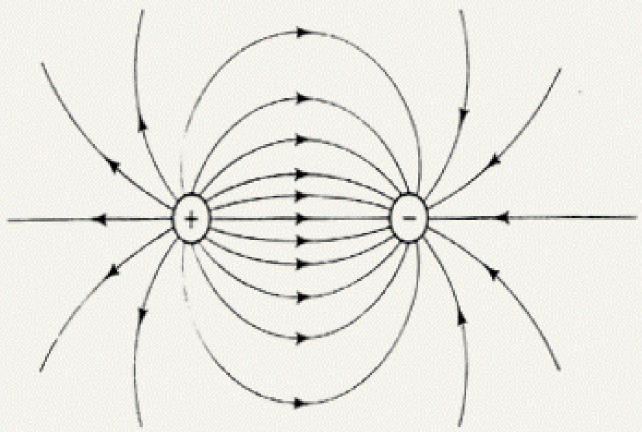
These vectors can be connected to form the *field lines.* You can imagine they are the paths of an infinitesimal test charge move under the force due to electric field.



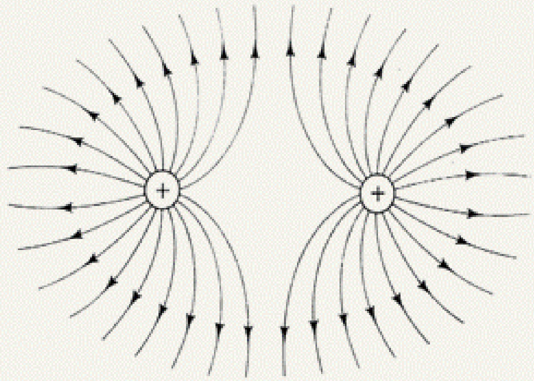
The magnitude of the field is indicated by the density of field lines.

Properties of field lines:

- Field lines emanate from a point charge symmetrically in all directions.
- Field lines begin on positive charges and end on negative charges (convention).
- They cannot simply stop in midair, though they may extend out to infinity.
- Field lines can never cross each other.

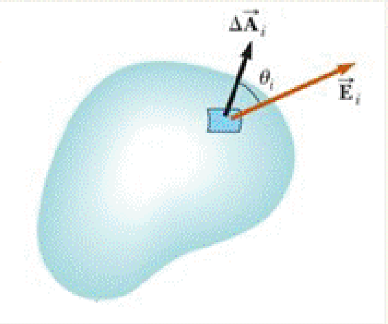


Equal but opposite charges



Equal charges

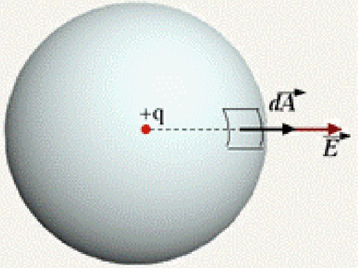
Gauss's Law:



Since in this model the field strength is proportional to the number of lines per unit area,

flux throgh a samll surface elemet $\Delta a_i = \Delta\Phi_i = E_i \Delta a_i \cos\theta$

$$= \boldsymbol{E}_i \cdot \Delta\boldsymbol{a}_i \qquad \Rightarrow \text{total flux} = \Phi = \sum_{\Delta a \to 0} \boldsymbol{E}_i \cdot \Delta\boldsymbol{a}_i = \oint \boldsymbol{E} \cdot d\boldsymbol{a}$$



The flux of E through a sphere of radius r is:

$$\oint \vec{E} \cdot d\vec{a} = \int \frac{1}{4\pi\varepsilon_0} \left(\frac{q}{r^2} \hat{r}\right) \cdot \left(r^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}\right) = \frac{1}{\varepsilon_0} q$$

i.e. total number of field lines crossing the sphere is $\frac{q}{\varepsilon_0}$, In general the flux through any surface enclosing the charge q is $\frac{q}{\varepsilon_0}$.

If there are more than one charge, according to the principle of superposition, the total field is the sum of all the individual fields:

Gauss's Law in integral form $\oint \boldsymbol{E} \cdot d\boldsymbol{a} = \sum (\frac{1}{\varepsilon_0} q_i) = \frac{Q_{enclosed}}{\varepsilon_0}$

according to the divergence theorem

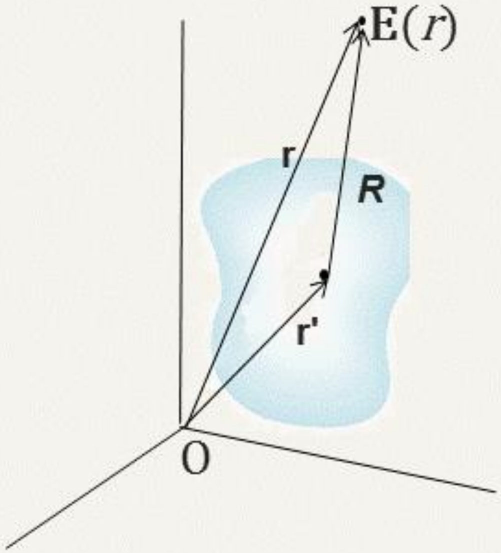
$$\oint_{surface} \vec{E} \cdot d\vec{a} = \int_{volume} (\nabla \cdot \vec{E}) d\tau = \frac{1}{\varepsilon_0} Q_{enc}$$

$$\text{but } \frac{1}{\varepsilon_0} Q_{enc} = \int_{volume} (\frac{1}{\varepsilon_0} \rho(r) d\tau \quad \Rightarrow \int_V (\nabla \cdot \vec{E}) d\tau = \int_V \frac{\rho(r)}{\varepsilon_0} d\tau$$

So the Gauss's law in differential form $\nabla \cdot \boldsymbol{E}(r) = \frac{1}{\varepsilon_0} \rho(r)$

This can be derived directly from the definition of electric field:

$$\vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int_{all\ space} \frac{\hat{R}}{R^2} \rho(\vec{r}')d\tau', \quad \boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$$



Taking divergence of E

$$\nabla \cdot \vec{E}(r) = \frac{1}{4\pi\varepsilon_0} \int_{\text{all space}} \nabla \cdot \left(\frac{\hat{R}}{R^2} \rho(r') \right) d\tau'$$

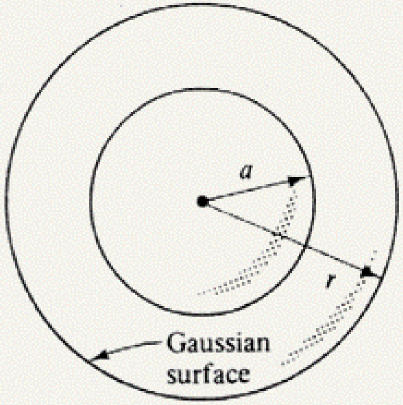
(the divergence calculated at r, r-dependence is contained in $R = \boldsymbol{r} - \boldsymbol{r}'$)

But $\nabla \cdot \left(\frac{\hat{R}}{R^2}\right) = 4\pi\delta^3(R)$

Thus $\nabla \cdot \boldsymbol{E} = \frac{1}{4\pi\varepsilon_0} \int 4\pi\delta^3(\boldsymbol{r} - \boldsymbol{r}')\rho(\boldsymbol{r}')d\tau' = \frac{1}{\varepsilon_0}\rho(r)$

2.2.3 Application of Gauss's Law

Example 1 Find the field outside a uniformly charged sphere of radius a



Sol:

$$\oint_{surface} \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

E point radially outward ,as does $d\mathbf{a}$

$$\oint_{surface} \mathbf{E} \cdot d\mathbf{a} = \int |E| da$$

(Gaussian Surface: surface through which the flux of a vector field is calculated)

E is constant over the spherical Gaussian surface

$$\oint_{surface} |E| da = |E| \oint_{surface} da = |E| 4\pi r^2$$

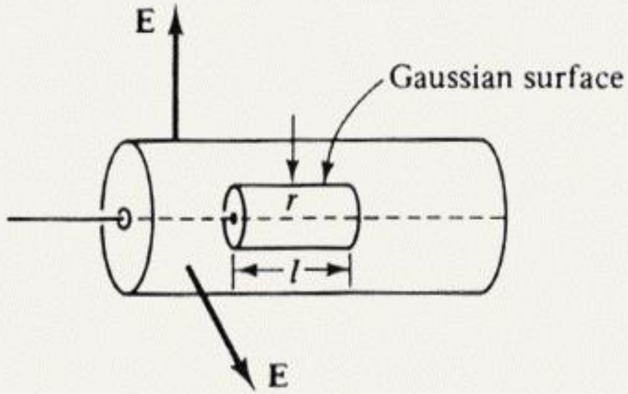
Thus $|E| 4\pi r^2 = \frac{1}{\varepsilon_0} q \implies E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$

What is the field inside the sphere?

Example 2: Find the electric field inside the cylinder which contains a charge density $\rho = kr$

Solution:

$$\oint_{surface} \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$



The enclosed charge is

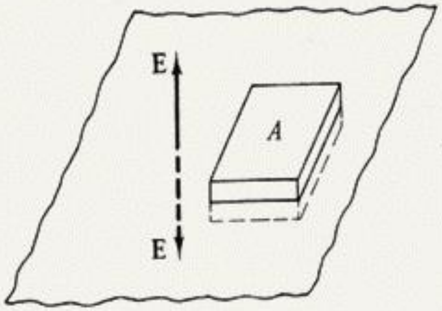
$$Q_{enc} = \int \rho \, d\tau = \int (kr')(r'dr'd\varphi \, dz) = 2\pi k l \int_0^r r'^2 dr' = \frac{2}{3}\pi k l r^3$$

$$\oint \vec{E} \cdot d\vec{a} = \int |E| da = |E| \int da = |E| 2\pi r l$$ (by symmetry)

thus $$|E| 2\pi r l = \frac{1}{\varepsilon_0} \frac{2}{3} \pi k l r^3 \quad \Rightarrow \quad \boxed{\vec{E} = \frac{1}{3\varepsilon_0} k r^2 \hat{r}}$$

Example 3: An infinite plane carries a uniform surface charge. σ
Find its electric field.

Solution: Draw a ”Gaussian pillbox
Apply Gauss’s law to this surface



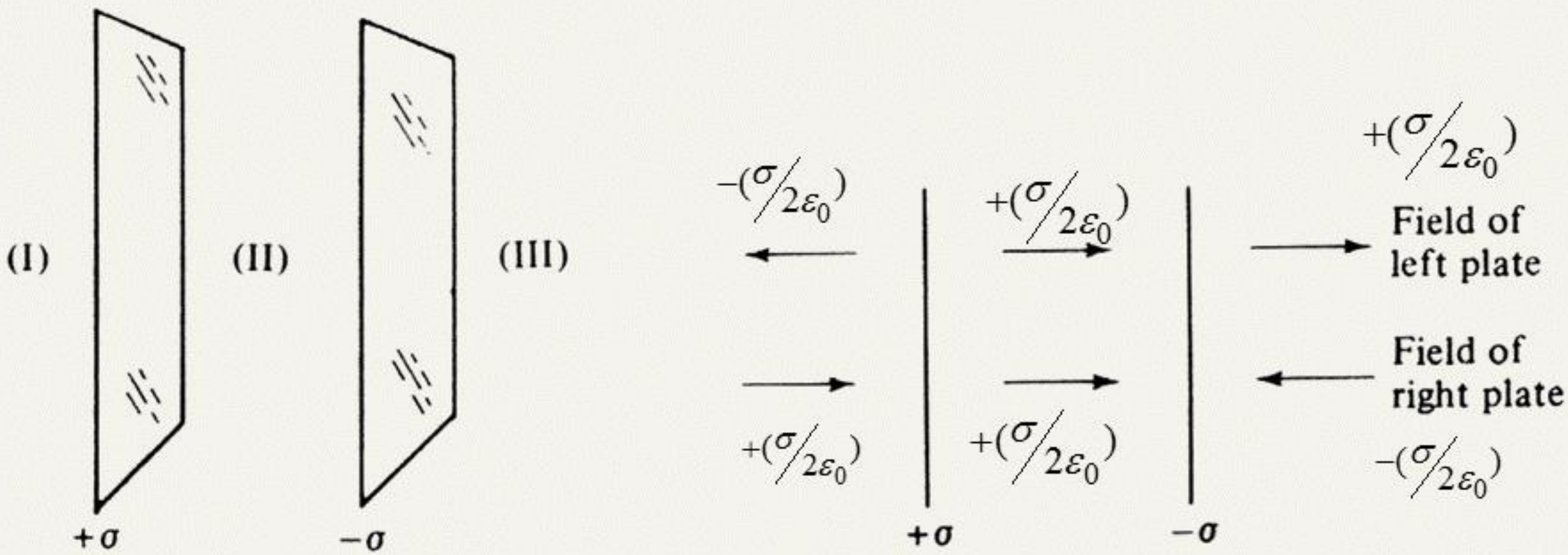
$$\oint_{surface} \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{enc}$$

By symmetry, E points away from the plane
thus, the top and bottom surfaces yields

$$\int \vec{E} \cdot d\vec{a} = 2A|E|$$ (there is no contribution from sides)

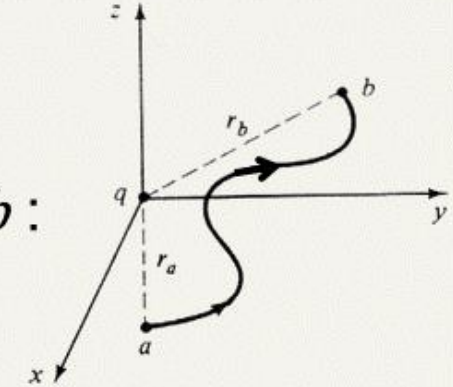
$$2A|E| = \frac{1}{\varepsilon_0} \sigma A \quad \Longrightarrow \quad \boxed{\vec{E} = \frac{\sigma}{2\varepsilon_0} \hat{n}}$$

Example 4 Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$ .Find the field in each of the three regions.



Field from each plate adds in between plates and cancels elsewhere .

The Curl of E



Calculate the line integral of the field from a point charge q at origin from some point a to another point b:

$$\int_a^b \vec{E} \cdot d\vec{l}$$

In spherical coordinate $$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}$$

$$d\vec{l} = dr\,\hat{r} + r d\theta\,\hat{\theta} + r\sin\theta d\varphi\,\hat{\varphi} \qquad \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} d\vec{r}$$

$$\int_a^b \vec{E} \cdot d\vec{l} = \frac{1}{4\pi\varepsilon_0} \int_a^b \frac{q}{r^2} dr = \frac{-1}{4\pi\varepsilon_0} \frac{q}{r}\Bigg|_{r_a}^{r_b} = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_a} - \frac{q}{r_b}\right)$$

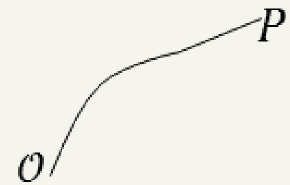
This integral is independent of path. It depends on the two end points $\Rightarrow$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

by Stokes' theorem $\Rightarrow$ $$\nabla \times \vec{E} = 0$$

Electrical potential

define a function: $V(P) = -\int_{\mathcal{O}}^{P} \boldsymbol{E} \cdot d\boldsymbol{l}$



Where $\mathcal{O}$ is some standard reference point ; V depends only on the point P. V is called the *electric potential*.

$$V(b) - V(a) = -\int_{\vartheta}^{b} \vec{E} \cdot d\vec{l} - \left(-\int_{\vartheta}^{a} \vec{E} \cdot d\vec{l} \right) = -\int_{a}^{b} \vec{E} \cdot d\vec{l}$$

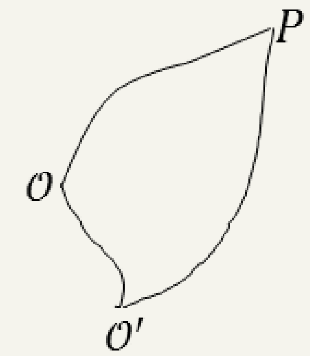
The fundamental theorem for gradients

$$V(b) - V(a) = \int_{a}^{b} (\nabla V) \cdot d\vec{l}$$

so $$\int_{a}^{b} (\nabla V) \cdot d\vec{l} = -\int_{a}^{b} \vec{E} \cdot d\vec{l} \Longrightarrow \boxed{\vec{E} = -\nabla V}$$

As expected (vector whose curl is zero is equal to the gradient of some scalar field) .

Reference point



Changing the reference point amounts to adds a constant to the potential

$$V'(p) = -\int_{O'}^{P} \boldsymbol{E}\cdot d\boldsymbol{l} = -\int_{O'}^{O} \boldsymbol{E}\cdot d\boldsymbol{l} + \int_{O}^{P} \boldsymbol{E}\cdot d\boldsymbol{l}$$

$$= \mathrm{K} + \int_{O}^{P} \boldsymbol{E}\cdot d\boldsymbol{l} = K + V(p)$$ (Where K is a constant)

Since the reference point is arbitrary, potential is determined only up to a constant. It does not affect the potential difference between two points:

$$V'(b) - V'(a) = V(b) - V(a)$$

Since the derivative of a constant is zero: $\nabla V = \nabla V'$ for different V, the field E remains the same.

Usually zero potential is set at infinity $V(\infty) = 0$

so $V(P) = -\int_{\infty}^{P} \boldsymbol{E}\cdot d\boldsymbol{l} = \int_{\boldsymbol{P}}^{\infty} \boldsymbol{E}\cdot d\boldsymbol{l}$

Potential obeys the superposition principle

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \cdots$$

Integrating from the common reference point to p ,

$$\int_{\infty}^{P} \mathbf{E} \cdot dl = \int_{\infty}^{P} \mathbf{E}_1 \cdot dl + \int_{\infty}^{P} \mathbf{E}_2 \cdot dl + \cdots$$

$$V = V_1 + V_2 + \cdots$$

Unit of potential:

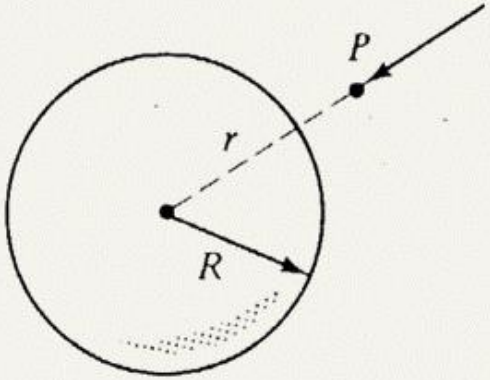
Unit of E : Newton/Coulomb

⇒ E.dl : Newton. Meter/Coulomb : Joule/Coulomb = Volt

This is also obvious from the fact that the electrical potential is also the amount of work done to bring a unit charge from infinity.

2.3.2

Example 2.6 Find the potential inside and outside a spherical shell of radius R, which carries a uniform surface charge (the total charge is q).



solution:

$$\vec{E}_{in} = 0 \qquad \vec{E}_{out} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

for r>R:

$$V(\vec{r}) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{\ell} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

for r≤R:

$$V(R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \qquad V(\vec{r}) \neq V(R) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$$

↑

$$\vec{E}_{in} = 0$$

2.3.3 Poisson's & Laplace's Equations for potential

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \quad \text{since } \vec{E} = -\nabla V \quad \Rightarrow \quad \nabla \cdot (-\nabla V) = -\nabla^2 V = \frac{\rho}{\varepsilon_0}$$

Poisson's Eq. $\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$

$\rho = 0$ $\quad \boxed{\nabla^2 V = 0}$ Laplace's eq.

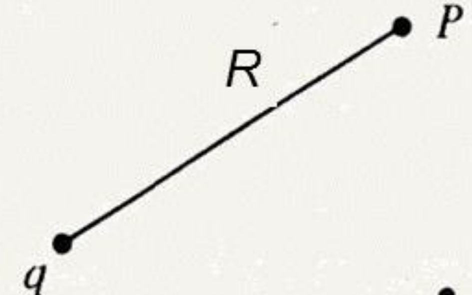
2.3.4 The Potential of a Localized Charge Distribution

$$\vec{E} = -\nabla V \qquad V = -\int_{\infty}^{r} E dr' \qquad V_{\infty} = 0$$

$$V(r) = \frac{-1}{4\pi\varepsilon_0}\int_{\infty}^{r}\frac{q}{r'^2}dr' = \frac{1}{4\pi\varepsilon_0}\frac{q}{r'}\bigg|_{\infty}^{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

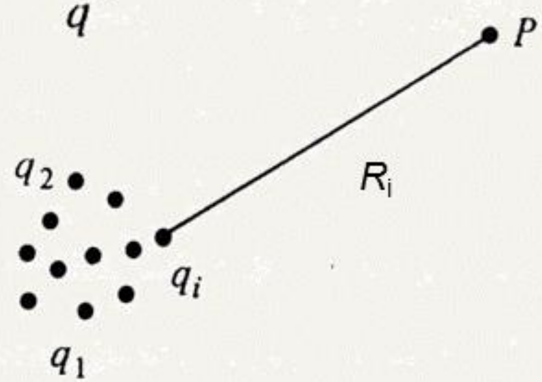
- Potential for a point charge

$$V(P) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R} \qquad R = \left|\vec{r} - \vec{r}_p\right|$$



- Potential for a collection of charge

$$V(P) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{R_i} \qquad R_i = \left|\vec{r}_i - \vec{r}_p\right|$$



Potential of a continuous distribution

for volume charge

$$\delta q = \rho d\tau$$

$$V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho}{R} d\tau$$

for a line charge

$$\delta q = \lambda d\ell$$

$$V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\lambda}{R} d\ell$$

for a surface charge

$$\delta q = \sigma da$$

$$V(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma}{R} da$$

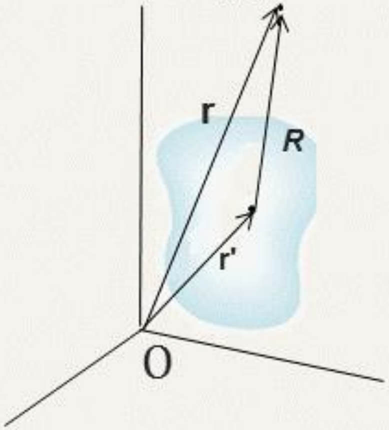
Corresponding electric field

$$\left(-\nabla \frac{1}{R} = \frac{\hat{R}}{R^2}\right)$$

$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{R^2} \rho \, d\tau$$

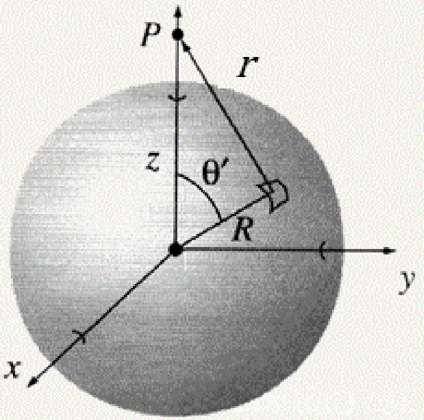
$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{R^2} \lambda d\ell$$

$$\vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{R^2} \sigma \, da$$



Example(2.8) : Find the potential of a uniformly charged spherical shell of radius R *(using above formulae).*

$$V(r) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma}{r}\, da', \qquad r^2 = R^2 + z^2 - 2Rz\cos\theta'$$



$$4\pi\varepsilon_0 V(z) = \sigma\int \frac{R^2 \sin\theta' d\theta' d\varphi'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}$$

$$= 2\pi R^2\sigma \int_0^{\pi} \frac{\sin\theta'}{\sqrt{R^2 + z^2 - 2Rz\cos\theta'}}\, d\theta'$$

$$= 2\pi R^2\sigma \left(\frac{1}{Rz}\sqrt{R^2 + z^2 - 2Rz\cos\theta'}\right)\Bigg|_0^{\pi}$$

$$= \frac{2\pi R\sigma}{z}\left(\sqrt{R^2 + z^2 + 2Rz} - \sqrt{R^2 + z^2 - 2Rz}\right)$$

$$= \frac{2\pi R\sigma}{z}\left[\sqrt{(R+z)^2} - \sqrt{(R-z)^2}\right]$$

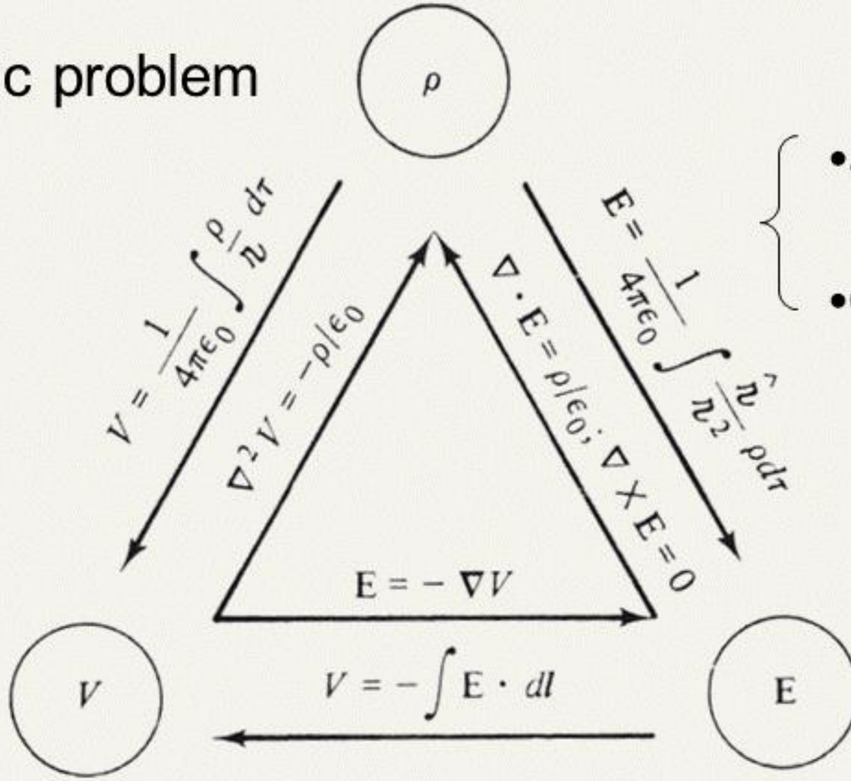
$$V(z) = \frac{R\sigma}{2\varepsilon_0 z}[(R+z) - (z-R)] = \frac{R^2\sigma}{\varepsilon_0 z} = \frac{q}{4\pi\varepsilon_0 r}, \quad \text{outside}$$

$$V(z) = \frac{R\sigma}{2\varepsilon_0 z}[(R+z) - (R-z)] = \frac{R\sigma}{\varepsilon_0} = \frac{q}{4\pi\varepsilon_0 R}, \quad \text{inside}$$

$\because$ total charge $q = 4\pi\sigma R^2$

2.3.5 Electrostatic Boundary Conditions

Electrostatic problem



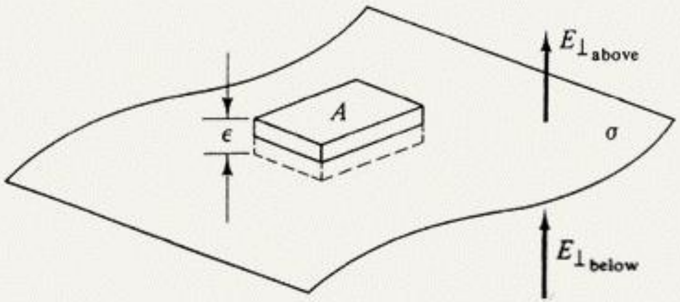
- Superposition
- Coulomb law

The above equations are differential or integral.
For a unique solution, we need boundary conditions. (e.g. , $V(\infty)=0$)
(boundary value problem in dynamics called *initial value problem*.)

Electric field B.C. at surface with charge (2.3.5(2)

Field normal (⊥) to surface:

⊥

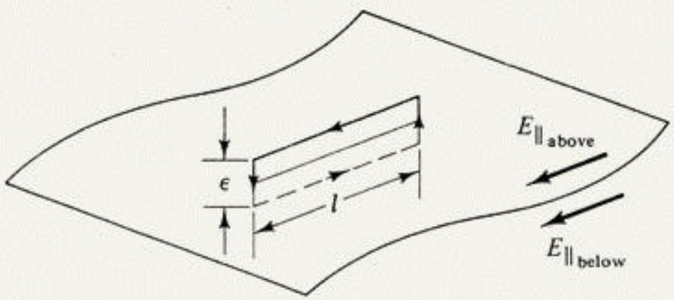


$$\oint_{surface} \vec{E}\cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0}$$

$$= E_{\perp above}\cdot A - E_{\perp below}\cdot A$$

$$E_{\perp above} - E_{\perp below} = \frac{\sigma}{\varepsilon_0}$$

Field parallel (∥) to surface:

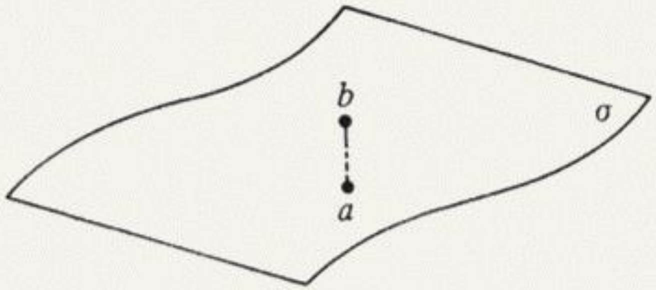


$$\oint \vec{E}\cdot d\vec{l} = 0 \quad \left(ie.\ \nabla\times\vec{E} = 0\right)$$

$$E_{\parallel\, above} = E_{\parallel\, below}$$

So in vector form: $$\boxed{\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}}$$

Potential boundary conditions (2.3.5 (3))



$$\because V_{above} - V_{below} = -\int_a^b \vec{E} \cdot d\vec{\ell} \xrightarrow{\overline{ab} \to 0} 0$$

$$\therefore \boxed{V_{above} = V_{below}}$$

$$E_{\perp above} - E_{\perp below} = \frac{\sigma}{\varepsilon_0} \qquad \vec{E} = -\nabla V$$

$$\therefore \boxed{\frac{\partial}{\partial n} V_{above} - \frac{\partial}{\partial n} V_{below} = -\frac{\sigma}{\varepsilon_0}}$$

Electromagnetic Theory

PHYS 401

Work and Energy in Electrostatics

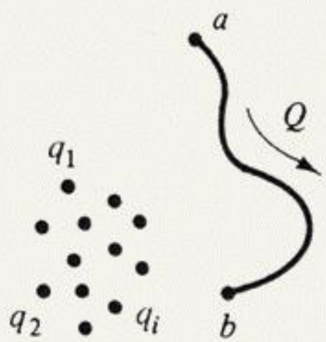
- The Work Done in Moving a Charge
- The Energy of a Point Charge Distribution
- The Energy of a Continuous Charge Distribution
- Comments on Electrostatic Energy

2.4.1 The Work Done in Moving a charge

In an electric field $\vec{E}$ a test charge Q feels a force $Q\vec{E}$

The total work done in moving from a to b

$$W = \int_a^b \vec{F} \cdot d\vec{\ell} = Q\int_a^b \vec{E} \cdot d\vec{\ell} = Q\left[V(b) - V(a)\right]$$



$$\boxed{V(b) - V(a) = \frac{W}{Q}}$$

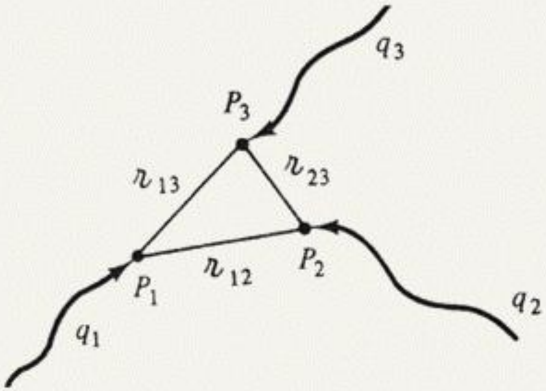
So, work done in bring a charge from ∞ to P:

$$W = Q\left[V(P) - V(\infty)\right] = QV(P)$$

$$V(\infty) = 0$$

2.4.2 The Energy of a Point Charge Distribution

It takes no work to bring in first charges



$W_1 = 0$ for q_1

Work needed to bring in q_2 is :

$$W_2 = q_2[\frac{1}{4\pi\varepsilon_0}\frac{q_1}{R_{12}}] = \frac{1}{4\pi\varepsilon_0}q_2(\frac{q_1}{R_{12}})$$

Work needed to bring in q_3 is :

$$W_3 = q_3[\frac{1}{4\pi\varepsilon_0}\frac{q_1}{R_{13}} + \frac{1}{4\pi\varepsilon_0}\frac{q_2}{R_{23}}] = \frac{1}{4\pi\varepsilon_0}q_3(\frac{q_1}{R_{13}} + \frac{q_2}{R_{23}})$$

Work needed to bring in q_4 is :

$$W_4 = \frac{1}{4\pi\varepsilon_0}q_4[\frac{q_1}{R_{14}} + \frac{q_2}{R_{24}} + \frac{q_3}{R_{34}}]$$

2.4.2

Total work

$$W = W_1 + W_2 + W_3 + W_4$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{R_{12}} + \frac{q_1 q_3}{R_{13}} + \frac{q_2 q_3}{R_{23}} + \frac{q_1 q_4}{R_{14}} + \frac{q_2 q_4}{R_{24}} + \frac{q_3 q_4}{R_{34}}\right)$$

$$= \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j>i}^{n}\frac{q_i q_j}{R_{ij}}$$

$$R_{ij} = R_{ji} \;\Rightarrow\; \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{q_i q_j}{R_{ij}} = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{R_{ij}}\right)$$

$$\boxed{W = \frac{1}{2}\sum_{i=1}^{n} q_i V(P_i)} \qquad V(P_i) = \sum_{\substack{j=1\\ j\neq i}}^{n}\frac{1}{4\pi\varepsilon_0}\frac{q_j}{R_{ij}}$$

2.4.3 The Energy of a Continuous Charge Distribution

$$W = \frac{1}{2}\int \rho V d\tau$$

$$\delta q = \rho \, d\tau$$

Volume charge density

$$\rho = \varepsilon_0 \nabla \cdot \vec{E} \qquad \nabla \cdot \left(\vec{E}V\right) = \left(\nabla \cdot \vec{E}\right)V + \vec{E}\cdot\left(\nabla V\right)$$

$$\frac{\varepsilon_0}{2}\int\left(\nabla \cdot \vec{E}\right)V d\tau = \frac{\varepsilon_0}{2}\left[\int \nabla \cdot \left(\vec{E}V\right)d\tau + \int \vec{E}\cdot\left(-\nabla V\right)d\tau\right]$$

$$(\vec{E} = -\nabla V)$$

$$= \frac{\varepsilon_0}{2}\left(\oint_{surface} V\vec{E}\cdot d\vec{a} + \oint_{volume} E^2 d\tau\right)$$

$$surface \to \infty \ \vec{E} \to 0 \ \Rightarrow \ W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau$$

2.4.3

Example 2.8 Find the energy of a uniformly charged spherical shell of total charge q and radius R

Sol.1 : $q = 4\pi R^2 \sigma$ $\qquad V = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}$

$$W = \frac{1}{2}\int \rho V d\tau = \frac{1}{2}\int \sigma V da = \frac{1}{2}\int \frac{q}{A}\frac{1}{4\pi\varepsilon_0}\frac{q}{R} da = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$$

Sol.2 : $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$ $\qquad E^2 = \frac{q^2}{(4\pi\varepsilon_0)^2 r^4}$

$$W = \frac{\varepsilon_0}{2}\int E^2 d\tau = \frac{\varepsilon_0}{2}\int_{\text{outside sphere}} \frac{q^2}{(4\pi\varepsilon_o)^2}\frac{1}{r^4}\left(r^2 \sin\theta \, d\theta \, d\varphi \, dr\right)$$

$$= \frac{q^2}{32\pi^2\varepsilon_0}\left[4\pi\int_R^\infty \frac{1}{r^2} dr\right] = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$$

2.4.4 Comments on Electrostatic Energy

i) Where is the energy stored? In charge or in field ?

Both are fine in ES. But, it is useful to regard the energy as being stored in the field

$$\varepsilon_0 \frac{E^2}{2} = \quad \text{Energy per unit volume}$$

ii) For a discrete point charge distribution Electrostatic energy in the field

$$W = \frac{\varepsilon_0}{2} \int_{all\ space} E^2 d\tau \quad \Rightarrow \quad \text{Energy} > 0$$

Potential energy of the charges:

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j>i}}^{n} \frac{q_i q_j}{r_{ij}}$$

which could be <0
ie: for two charges q, -q

$$W = \frac{1}{4\pi\varepsilon_0} \frac{-q^2}{r_{12}}$$

ii) For a discrete point charge distribution Electrostatic energy in the field

$$W = \frac{\varepsilon_0}{2}\int_{all\ space} E^2 d\tau \quad \Rightarrow \quad \text{Energy} \quad > 0$$

Potential energy of the charges:

$$W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j>i}}^{n}\frac{q_i q_j}{r_{ij}}$$

which could be <0
ie: for two charges q, -q

$$W = \frac{1}{4\pi\varepsilon_0}\frac{-q^2}{r_{12}}$$

This contradiction is because, in

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V_i = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{q_i q_j}{r_{ij}}$$

potential of q_i due to its own field is not considered, i.e. the energy needed to create the charge q_i, which is infinite for a point charge.

$$\frac{1}{8\pi\varepsilon_0}\frac{q^2}{R} \rightarrow \infty \text{ as } R \rightarrow \infty$$

In quantum electrodynamics this is solved by renormalizing, for classical EM we will ignore this self of point charge.

(iii)The superposition principle does not apply for ES energy

$$W_1 = \frac{\varepsilon_0}{2}\int E_1^2 d\tau \qquad W_2 = \frac{\varepsilon_0}{2}\int E_2^2 d\tau$$

$$W_{tot} = \frac{\varepsilon_0}{2}\int (\vec{E}_1 + \vec{E}_2)^2 d\tau$$

$$= \frac{\varepsilon_0}{2}\int (E_1^2 + E_2^2 + 2\vec{E}_1 \cdot \vec{E}_2) d\tau$$

$$= W_1 + W_2 + \varepsilon_0 \int (\vec{E}_1 \cdot \vec{E}_2) d\tau$$

2.5 Conductors

2.5.1 Basic Properties of Conductors

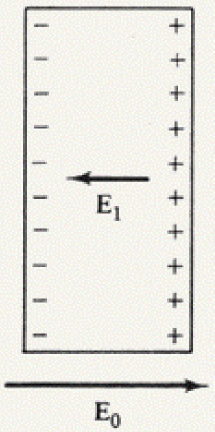
2.5.2 Induced Charges

2.5.3 The Surface Charge on a Conductor;
the Force on a Surface Charge

2.5.4 Capacitors

2.5.1 Basic Properties of Conductors

Charges are free to move in a conductor, a perfect conductor has an infinite amount of free chargers, metals come pretty close.



$\vec{E} = 0$

(1) inside a conductor electric field is zero
otherwise, the free charges will move to make $\vec{E} \longrightarrow 0$
inside the conductor

(2) $\rho = 0$ inside a conductor

$$\because \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} \qquad \vec{E} = 0 \quad \Rightarrow \quad \rho = 0$$

(3) Any net charge resides on the surface

(4) Potential is constant throughout a conductor (equipotential).

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot d\vec{l} = 0 \quad \Rightarrow V(b) = V(a)$$

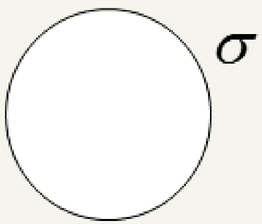
2.5.1 (2)

(5) Electric field just outside the conductor is perpendicular to the surface

Otherwise, $\vec{E}_{\parallel}$ will move the free charge to make $\vec{E}$ perpendicular to the surface, just outside a conductor

Free charges staying on the surface have the a minimum energy (charge in a conductor distribute to a configuration that minimize the energy like any other free dynamic system).

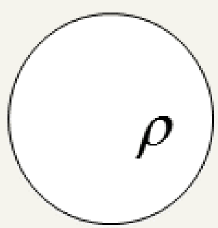
e.g.



$$Energy = E_s = \frac{1}{8\pi\varepsilon_0}\frac{q^2}{R}$$

$E_{in} = 0$ $\quad W_{in} = 0$

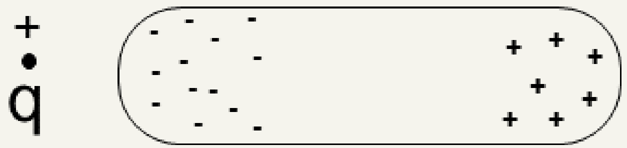
For a volume distribution



$$Energy = E_v = \frac{3}{20\pi\varepsilon_0}\frac{q^2}{R} \qquad E_v > E_s$$

$E_{in} \neq 0$ $\quad W_{in} \neq 0$

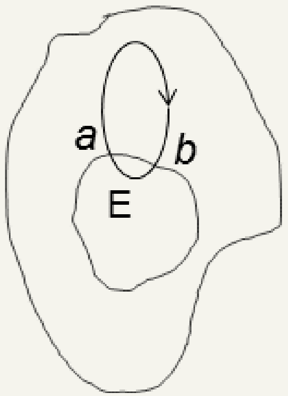
2.5.2 Induced Charges



Charge **q** pulls minus charges in the conductor towards it and push plus charges away.

This induced charge distribution is also neutralize the electric field inside the conductor.

Inside a conductive cavity electric field is zero.
Faraday cage: shield any external electric fields.



$$\oint \vec{E} \cdot d\vec{l} = 0$$

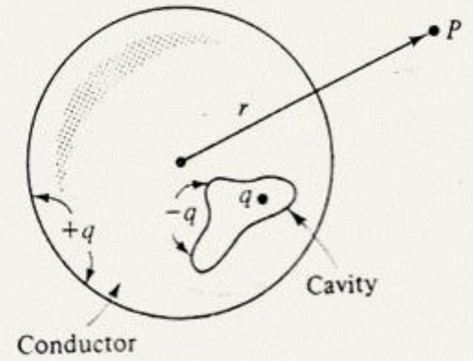
$$\int_b^a \vec{E} \cdot d\vec{l} = 0 \qquad a,b \text{ are arbitrary chosen}$$

in cavity

$$\Rightarrow \vec{E}_{in\ cavity} = 0$$

2.5.2 Induced Charge

Example 2.9 A spherical conductor has a cavity of irregular shape. If a charge q is placed inside the cavity what is the field outside the sphere?



Charge -q induced on the cavity surface distributes to shield q and to make $\vec{E}=0$ in the conductor

from charge conservation and symmetry, induced +q *charge* uniformly distributes on the surface

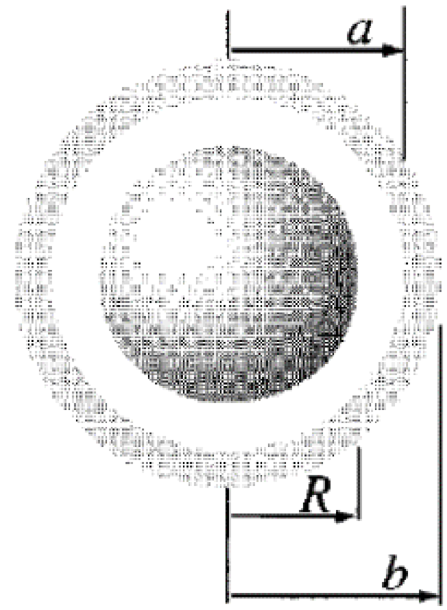
$$\therefore \ \vec{E}(P) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$

Problem 2.35 A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b, as in Fig. 2.48). The shell carries no net charge.

(a) Find the surface charge density σ at R, at a, and at b.

(b) Find the potential at the center, using infinity as the reference point.

(c) Now the outer surface is touched to a grounding wire, which lowers its potential to zero (same as at infinity). How do your answers to (a) and (b) change?



2.5.3 The Surface Charge on a Conductor

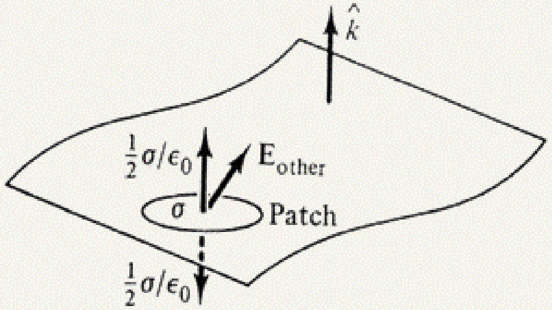
$$\vec{E}_{above} - \vec{E}_{below} = \frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\because E_{in} = 0 \ \text{ or } \ \vec{E}_{below} = 0$$

$$\therefore -\nabla V = \vec{E}_{above} = \frac{\sigma}{\varepsilon_0}\hat{n}$$ or $$\boxed{\frac{\partial V}{\partial n} = -\frac{\sigma}{\varepsilon_0}}$$

if we know V or E, we can get σ.

Force on a surface charge:



Due to surface charge electric field is different above and below the surface.
Take a small patch on the surface, The field due to change on the patch above /below, very close to the surface is $\frac{\sigma}{2\epsilon_0}\hat{k}$,

Field due to charges outside the patch = $\vec{E}_{other}$

$$\vec{E}_{above} = \vec{E}_{other} + \frac{\sigma}{2\varepsilon_0}\hat{k} \qquad \vec{E}_{below} = \vec{E}_{other} - \frac{\sigma}{2\varepsilon_0}\hat{k}$$

$$\vec{E}_{other} = \frac{1}{2}(\vec{E}_{above} + \vec{E}_{below}) \quad = \quad \vec{E}_{average}$$

Force on the charge in the patch $\sigma\vec{E}_{other} = \sigma\vec{E}_{average}$.

In case of a conductor field inside is zero, and $\frac{\sigma}{\epsilon_0}\hat{n}$ outside

$$f = \frac{1}{2}\sigma\vec{E}_{above} = \frac{\sigma^2}{2\varepsilon_0}\hat{n}$$

electrostatic pressure $P = \dfrac{\sigma^2}{2\varepsilon_0} = \dfrac{\varepsilon_0}{2}E^2$

Problem 2.37 Two large metal plates (each of area A) are held a distance d apart. Suppose we put a charge Q on each plate; what is the electrostatic pressure on the plates?

Problem 2.38 A metal sphere of radius R carries a total charge Q. What is the force of repulsion between the "northern" hemisphere and the "southern" hemisphere?

2.5.4 Capacitors

Consider 2 conductors (Fig 2.53)

The potential difference

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \vec{E} \cdot d\vec{l} \qquad (V \text{ is constant.})$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\int \frac{\hat{r}}{r^2}\rho d\tau$$ double $\rho \rightarrow$ double $Q \rightarrow$ double $\vec{E} \rightarrow$ double V

Define the ratio between Q and V to be capacitance

$$C = \frac{Q}{V}$$ a geometrical quantity

in mks 1 farad(F)= 1 Coulomb / volt

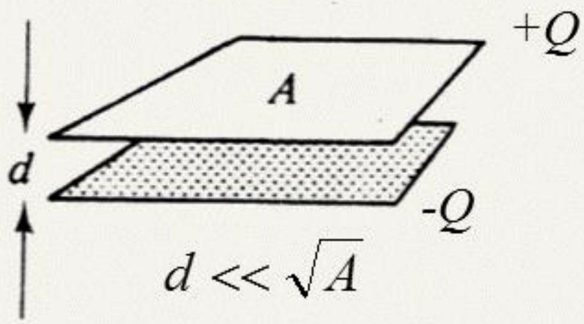
↖ inconveniently large ;

$10^{-6} F$: *microfarad*

$10^{-12} F$: *picofarad*

2.5.4 (2)

Example: Find the capacitance of a “parallel-plate capacitor”?



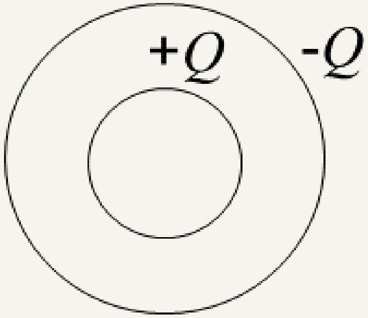
$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0}$$

$$V = E \cdot d = \frac{Q}{A\varepsilon_0} d$$

$$C = \frac{A\varepsilon_0}{d}$$

2.5.4 (3)

Example: Find capacitance of two concentric spherical shells with radii a and b .



$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$$

$$V = -\int_b^a \vec{E}\cdot d\vec{l} = -\frac{Q}{4\pi\varepsilon_0}\int_b^a \frac{1}{r^2}dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$C = \frac{Q}{V} = 4\pi\varepsilon_0\frac{ab}{(b-a)}$$

2.5.4 (4)

The work to charge up a capacitor

$$dW = Vdq = (\frac{q}{C})dq$$

$$W = \int_0^Q (\frac{q}{C})dq = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV = \frac{1}{2}CV^2$$

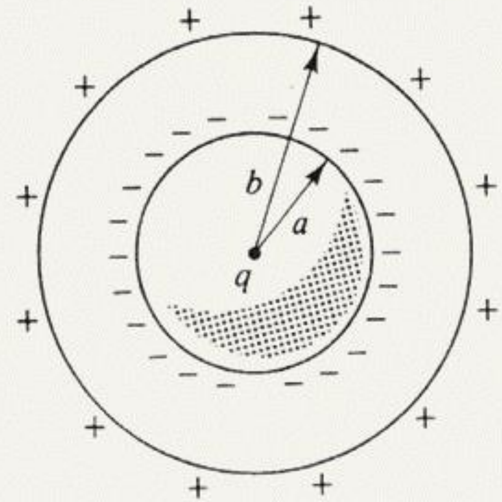
2.5.1 (3)

Example : A point charge q at the center of a spherical conducting shell. How much induced charge will accumulate there?

Solution :

$$\because E_{in} = 0 \qquad q \text{ induced}$$

$$\therefore 4\pi a^2 \cdot \sigma_a = -q \qquad \sigma_a = -\frac{1}{4\pi}\frac{q}{a^2}$$



charge conservation

$$4\pi b^2 \cdot \sigma_b = -4\pi a^2 \sigma_a \qquad \sigma_b = \frac{1}{4\pi}\frac{q}{b^2}$$

$$E_{in} = 0$$

$$Q_{enc} = q + q_{induced} = 0$$

$$q_{induecd} = -q$$