Vector Analysis

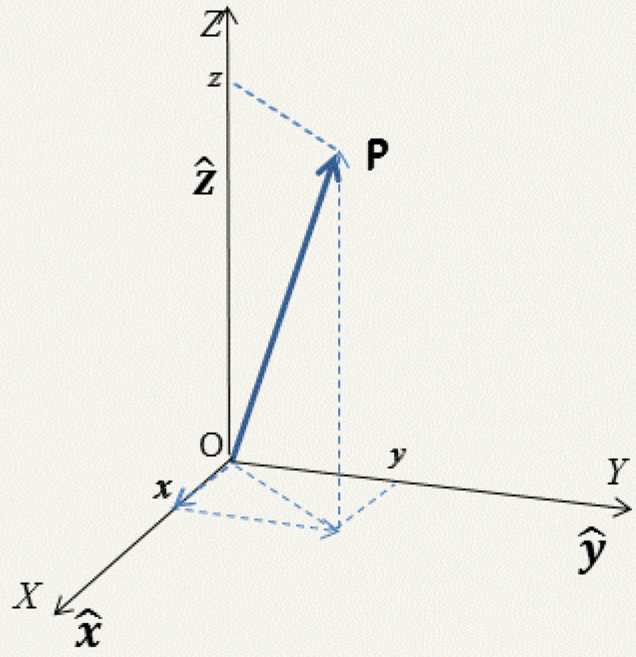
Electromagnetic Theory
PHYS 401

Fall 2018

Coordinate Systems

Cartesian coordinates :

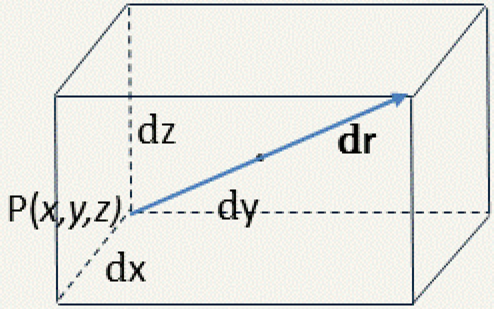
- Three mutually orthogonal axes X,Y,Z, unit vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$ arein the direction of increasing coordinate value.
  A point P in space is given by the projections x, y, z on coordinate axes.



$$P(x, y, z) = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

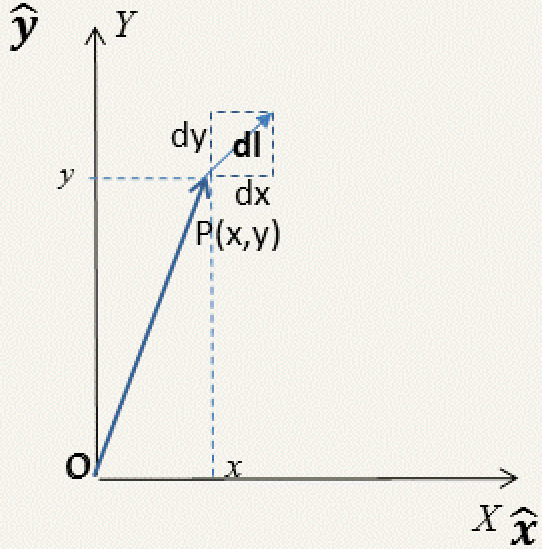
$$-\infty < x < \infty, -\infty < y < \infty, -\infty < z < \infty$$

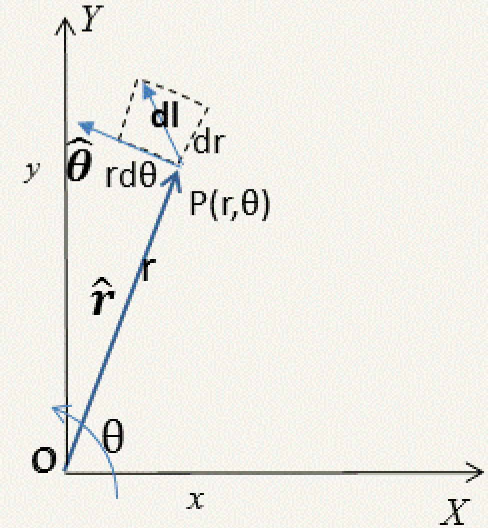
Infinitesimal volume element = $dxdydz$



Infinitesimal line element $d\boldsymbol{r} = dx\,\hat{\boldsymbol{x}} + dy\,\hat{\boldsymbol{y}} + dz\,\hat{\boldsymbol{z}}$

Polar coordinates (2-dimensions):





In 2D Cartesian coordinates position (x, y)

infinitesimal line elements: $dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}}$

infinitesimal area element: $dxdy$

In polar coordinates: position (r, θ)

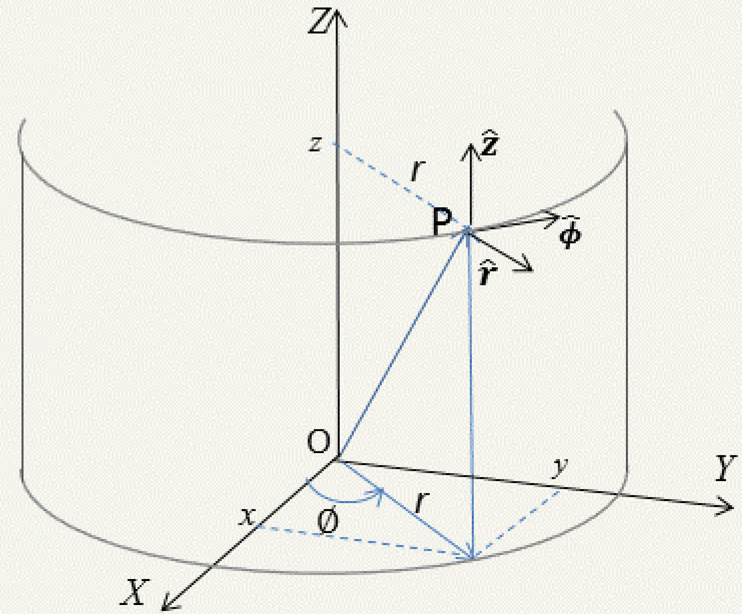
infinitesimal increments line elements: $\hat{\boldsymbol{r}}dr + \hat{\boldsymbol{\theta}}rd\theta$

infinitesimal area element : $rdrd\theta$

e.g. area of a circle: $\int_0^R \int_0^{2\pi} rdrd\theta = \pi R^2$

Unlike in Cartesian system not all unit vectors are fixed. Directions of $\hat{\boldsymbol{r}}$ and $\hat{\theta}$ are depend on the position (polar angle θ).

Cylindrical Coordinates



- In cylindrical coordinates position of a point P is given by:
  - r : the radial distance from Y axis
  - $\emptyset$: the azimuthal angle, measured from the X-axis in the XY plane
  - z : the distance from the XY plane (same as in the Cartesian system)

$$0 < r < \infty, \;\; 0 < \phi < 2\pi, \;\; -\infty < z < \infty$$

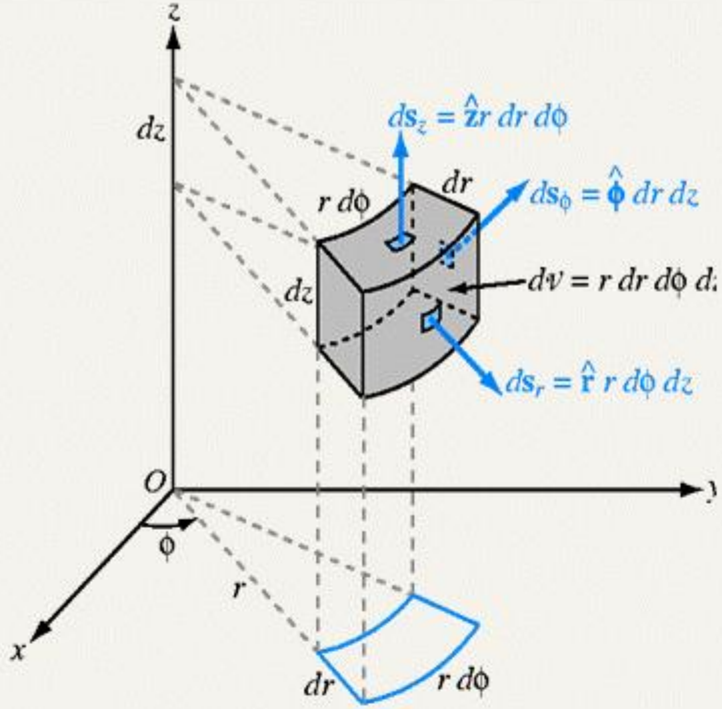
- Unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\boldsymbol{z}}$ are in the direction of increasing coordinate values.
- Directions of $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\phi}}$ are depend on the position (azimuthal angle ϕ).

Relation between Cartesian and cylindrical coordinates.

$$x = r\cos\phi\,,\; y = \mathrm{r}\;\sin\phi,\; z = z$$

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}}\cos\boldsymbol{\phi} + \hat{\boldsymbol{y}}\sin\boldsymbol{\phi}\;,\;\; \hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\sin\boldsymbol{\phi} + \hat{\boldsymbol{y}}\cos\boldsymbol{\phi}$$

$$\hat{\mathrm{x}} = \hat{\mathrm{r}}\cos\varphi - \hat{\varphi}\sin\varphi, \quad \hat{\mathrm{y}} = \hat{\mathrm{r}}\sin\varphi + \hat{\varphi}\cos\varphi$$



- Sides of the infinitesimal volume element: dz ; dr ; $r\, d\phi$
- infinitesimal volume element= $rdsdzd\phi$
- Del operator:

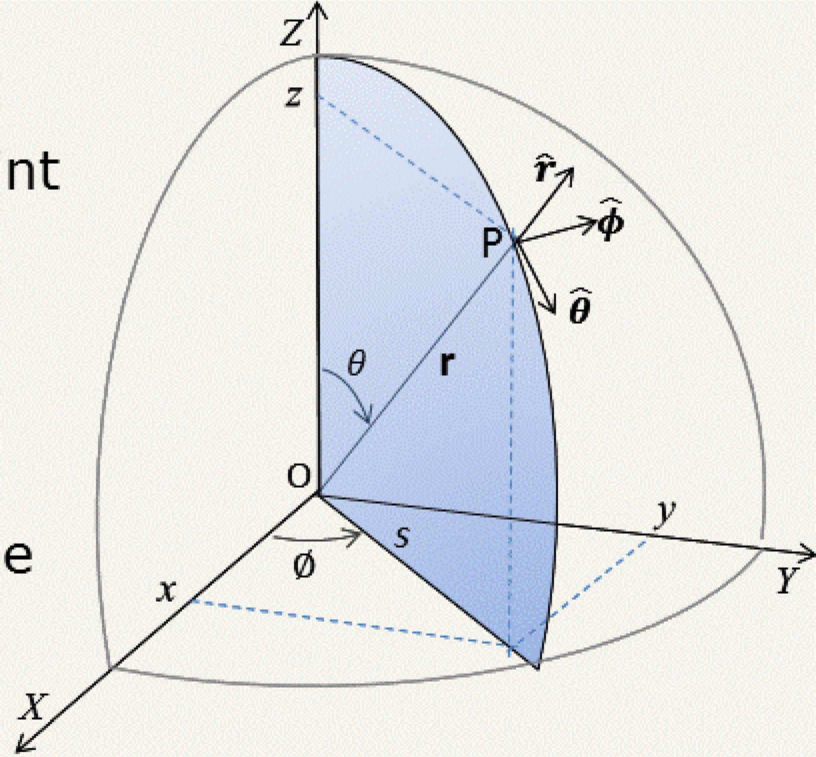
$$\nabla \varphi = \hat{r}\frac{\partial \varphi}{\partial r} + \hat{\varphi}\frac{1}{r}\frac{\partial \varphi}{\partial \varphi} + \hat{z}\frac{\partial \varphi}{\partial z}$$

$$\nabla . \mathbf{A} = \frac{1}{r}\frac{\partial (rA_s)}{\partial r} + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

Spherical Coordinates

In spherical coordinates position of a point P is given by:

- r: radial distance the origin
- $\emptyset$: azimuthal angle, measured from the X-axis in the XY plane
- θ : angle between the Z axis and the line from origin to point P



$$0 < r < \infty, \ \ 0 < \phi < 2\pi, \ \ 0 < \theta < \pi$$

- Unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\phi}}$, $\hat{\boldsymbol{\theta}}$ are in the direction of increasing coordinate values. Their directions depend on the position.

Relation between Cartesian and spherical coordinates.

$$x = r\sin\theta\cos\phi\,, \ \ y = r\sin\theta\sin\phi\,, \ \ z = r\cos\theta$$

$$\hat{\boldsymbol{r}} = \hat{\boldsymbol{x}}\sin\theta\cos\phi + \hat{\boldsymbol{y}}\sin\theta\sin\phi + \hat{\boldsymbol{z}}\cos\theta$$

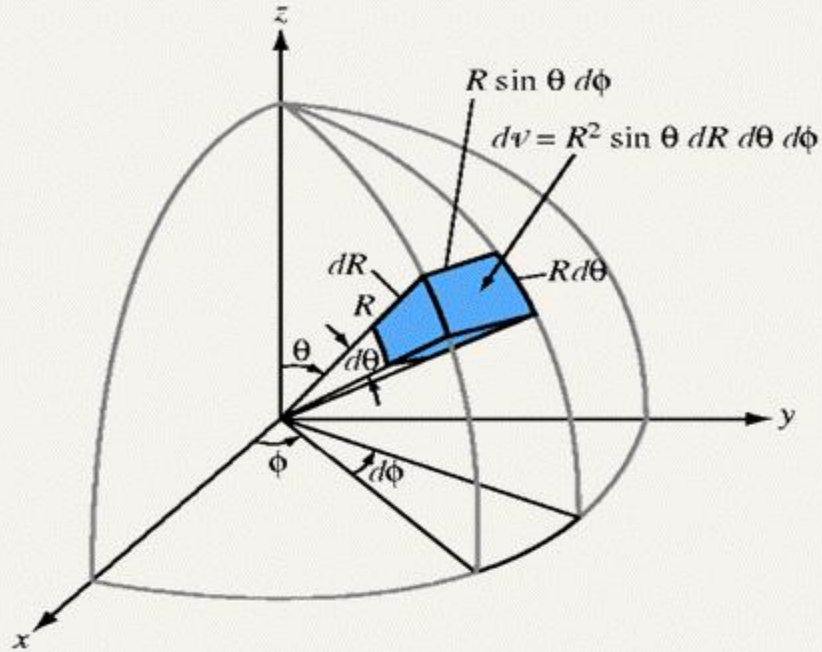
$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{x}}\cos\theta\cos\phi + \hat{\boldsymbol{y}}\cos\theta\sin\phi - \hat{\boldsymbol{z}}\sin\theta$$

$$\hat{\boldsymbol{\phi}} = -\hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi$$

$$\hat{\boldsymbol{x}} = \hat{\boldsymbol{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$$

$$\hat{\boldsymbol{y}} = \hat{\boldsymbol{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$$

$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\phi$$



Line element: $= dr\hat{\boldsymbol{r}} + r d\theta\hat{\boldsymbol{\theta}} + r\sin\theta d\phi\hat{\boldsymbol{\phi}}$

Volume element: $r^2\sin\theta d\theta d\phi$

Del operator :
$$\nabla\varphi = \hat{\boldsymbol{r}}\frac{\partial}{\partial r}\phi + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta}\varphi + \hat{\boldsymbol{\varphi}}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\varphi$$

$$\nabla\cdot\vec{V} = \frac{1}{r^2}\frac{\partial\left(r^2 V_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta V_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi}$$

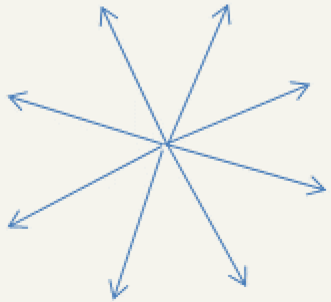
Example: A sphere of radius 2 cm contains a volume charge density ρ given by $10\rho cos^2\theta$ (Cm^{-3})Find the total charge Q contained in the sphere.

Summary

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Base vectors properties	$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}=\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}=1$ $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}=\hat{\mathbf{y}}\cdot\hat{\mathbf{z}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{x}}=0$ $\hat{\mathbf{x}}\times\hat{\mathbf{y}}=\hat{\mathbf{z}}$ $\hat{\mathbf{y}}\times\hat{\mathbf{z}}=\hat{\mathbf{x}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{x}}=\hat{\mathbf{y}}$	$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}=\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}=1$ $\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}}=\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}=0$ $\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}=\hat{\mathbf{z}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}}=\hat{\mathbf{r}}$ $\hat{\mathbf{z}}\times\hat{\mathbf{r}}=\hat{\boldsymbol{\phi}}$	$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}}=\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}}=1$ $\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\phi}}=\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{R}}=0$ $\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\phi}}$ $\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\phi}}=\hat{\mathbf{R}}$ $\hat{\boldsymbol{\phi}}\times\hat{\mathbf{R}}=\hat{\boldsymbol{\theta}}$
Dot product, $\mathbf{A}\cdot\mathbf{B}=$	$A_xB_x+A_yB_y+A_zB_z$	$A_rB_r+A_\phi B_\phi+A_zB_z$	$A_RB_R+A_\theta B_\theta+A_\phi B_\phi$
Cross product, $\mathbf{A}\times\mathbf{B}=$	$\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$	$\begin{vmatrix}\hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}}\\ A_r & A_\phi & A_z\\ B_r & B_\phi & B_z\end{vmatrix}$	$\begin{vmatrix}\hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}}\\ A_R & A_\theta & A_\phi\\ B_R & B_\theta & B_\phi\end{vmatrix}$
Differential length, $d\mathbf{l}=$	$\hat{\mathbf{x}}\,dx+\hat{\mathbf{y}}\,dy+\hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr+\hat{\boldsymbol{\phi}}r\,d\phi+\hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR+\hat{\boldsymbol{\theta}}R\,d\theta+\hat{\boldsymbol{\phi}}R\sin\theta\,d\phi$
Differential surface areas	$d\mathbf{s}_x=\hat{\mathbf{x}}\,dy\,dz$ $d\mathbf{s}_y=\hat{\mathbf{y}}\,dx\,dz$ $d\mathbf{s}_z=\hat{\mathbf{z}}\,dx\,dy$	$d\mathbf{s}_r=\hat{\mathbf{r}}r\,d\phi\,dz$ $d\mathbf{s}_\phi=\hat{\boldsymbol{\phi}}\,dr\,dz$ $d\mathbf{s}_z=\hat{\mathbf{z}}r\,dr\,d\phi$	$d\mathbf{s}_R=\hat{\mathbf{R}}R^2\sin\theta\,d\theta\,d\phi$ $d\mathbf{s}_\theta=\hat{\boldsymbol{\theta}}R\sin\theta\,dR\,d\phi$ $d\mathbf{s}_\phi=\hat{\boldsymbol{\phi}}R\,dR\,d\theta$
Differential volume, $d\mathcal{V}=$	$dx\,dy\,dz$	$r\,dr\,d\phi\,dz$	$R^2\sin\theta\,dR\,d\theta\,d\phi$

Delta function:

$$\boldsymbol{V} \equiv \frac{\hat{r}}{r^2}$$



Naive calculation gives:

$$\nabla\bullet\boldsymbol{V} = \left(\frac{1}{r^2}\frac{\partial\left(r^2 v_r\right)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial\left(\sin\theta v_\theta\right)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\left(v_\phi\right)}{\partial\phi} \right)$$

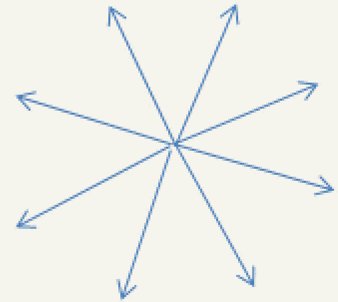
$$\nabla\bullet\boldsymbol{V} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

but it leads to contradiction with the divergence theorem , say applied over a sphere:

$$\int_{Surface} \vec{V}\bullet\hat{n}da = \int_{Surface} \frac{1}{r^2}r^2\sin\theta d\theta d\phi = \int_{Surface} d\Omega = 4\pi \neq \int_{volume} \nabla\cdot\vec{V}d\tau = 0$$

Problem is the field is ∞ at $r=0$, and is not correctly expressed

$$\boldsymbol{V} \equiv \frac{\hat{r}}{r^2}$$



Problem is the field is ∞ at r=0, and is not correctly expressed

According to the divergence theorm:

$$\int_{volume} \nabla \cdot \vec{V} d\tau = \int_{Surface} \vec{V} \bullet \hat{n} da = \int_{Surface} \frac{1}{r^2} r^2 d\Omega = 4\pi$$

which is independent of the radius of the sphere R, centered at the origin.

since $\nabla \cdot \vec{V} = 0$ except at r=0,

entire contribution to the intergal is from $r = 0$

so $\nabla \cdot \vec{V} = 0$ for $r \neq 0$

and $\int \nabla \cdot \vec{V} d\tau = 4\pi$ for any volume containing $r = 0$

Not an ordinary fuction!

Delta Function

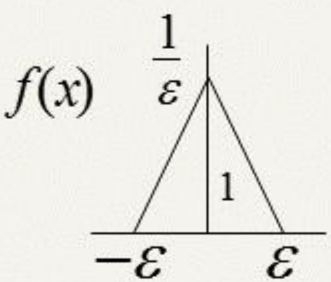
To work with such situations the Dirac delta function is $\delta(x)$ used:

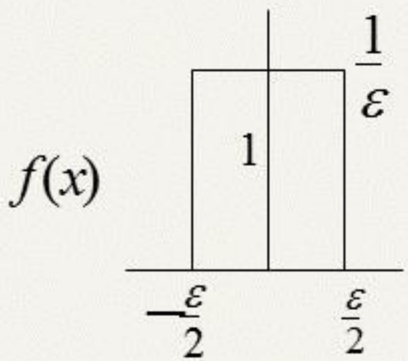
In 1-dimension it is defined as:

$$\delta(x) = 0 \text{ for x} \neq 0;\ \delta(0) = \infty \text{ such that } \int_{-\infty}^{\infty} \delta(x) dx = 1$$

It is an even fiction $\delta(-x) = \delta(x)$ with unit area $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$

Can be considered as the limit of functions:





$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon\sqrt{\pi}} e^{-(\frac{x}{\varepsilon})^2}$$

$$\lim_{\varepsilon \to 0} f(x) = \delta(x)$$

$\delta(x-a)=0$ for $x \neq a$ and $\delta(x-a)=\infty$ with $\int_{-\infty}^{\infty} \delta(x-a)=1$

for a function $f(x)$: $f(x)\delta(x)=f(0)\delta(x)$, $\int_{-\infty}^{\infty} f(y)\delta\left(x-y\right)dy=f(x)$

Other properties of the delta function:

1) $\delta\left(ax\right)=\frac{1}{|a|}\delta\left(x\right)$

2) $\delta\left(f(x)\right)=\left|\frac{df(x)}{dx}\right|^{-1}\delta\left(x\right)$

where $f(x_0)=0$

3) $x\frac{d}{dx}\left(\delta\left(x\right)\right)=-\delta\left(x\right)$

(4) $\delta\left(x-x_1\right)\delta\left(x-x_2\right)=\frac{\delta\left(x-x_1\right)+\delta\left(x-x_2\right)}{|x_1-x_2|}$

(5) $\delta\left(x^2-a^2\right)=\frac{1}{2a}\left(\delta\left(x-a\right)+\delta\left(x+a\right)\right)$

(6) $\delta\left(t-x\right)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-x)}d\omega$

3D Delta function

1D delta function can be generalized to 3 D as $\delta(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$

$$\int_{all\ space} \delta^3(\mathbf{r})dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\delta(y)\delta(z)dxdydz = 1$$

and $\int_{all\ space} f(\mathbf{r})\delta^3(\mathbf{r}-\mathbf{a})d\tau = f(\mathbf{a})$

Now consider $\boldsymbol{V} \equiv \frac{\hat{r}}{r^2}$

Since according to divergence theorem $\int_{vol} \frac{1}{4\pi} \nabla \cdot \vec{V}\, dv = 1$ and

$\nabla \cdot V = 0$ for $r \neq 0$, $\nabla \cdot V$ has the same properties as the delta function

$$\Rightarrow \frac{1}{4\pi} \nabla \cdot \boldsymbol{V} = \delta^3(\mathbf{r}) \Rightarrow \nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3(\mathbf{r})$$

$$\because \nabla\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2} \quad \Rightarrow \nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^3(\vec{r})$$

now

$$\int_{Sphere} \nabla \bullet \boldsymbol{v} d\tau = \int_{Sphere} \nabla \bullet \left(\frac{\hat{r}}{r^2} \right) d\tau = \int_{Sphere} 4\pi \delta^3 (\boldsymbol{r}) d\tau = 4\pi$$

$$= \int_{Surface} \vec{V} \bullet \hat{n} da$$

As required by the divergence theorem

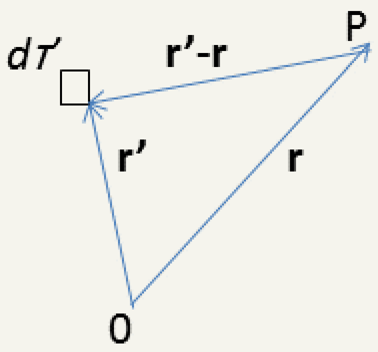
Helmholtz Theorem

(The fundamental theorem of vector analysis)

The Helmholtz theorem states that any continuous vector field can be written as a sum of a gradient of a scalar field and a curl of a vector field.

$$\vec{V}(\mathbf{r}) = \nabla U(r) + \nabla \times \vec{W}(\mathrm{r})$$

U is called the scalar potential and $\vec{W}$ is called the vector potential of the field **Proof:**



using $\nabla^2\vec{V} = \nabla(\nabla\cdot\vec{V}) - \nabla\times(\nabla\times\vec{V})$

define $\mathrm{U(r)} = \dfrac{1}{4\pi}\displaystyle\int_V \dfrac{\vec{V}(\boldsymbol{r}')}{|\boldsymbol{r}-\boldsymbol{r}'|}d\tau'$ where the volume V is all space

then $\nabla^2 U(r) = \dfrac{1}{4\pi}\displaystyle\int_V \vec{V}(\boldsymbol{r}')\nabla^2\dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}d\tau'$ (∇ applies on r)

$$= \frac{1}{4\pi}\int_V \vec{V}(\boldsymbol{r}')\delta(r-r')d\tau' = \vec{V}(r)$$

so $\vec{V}(r) = \nabla(\nabla\cdot\vec{V}(\boldsymbol{r})) - \nabla\times(\nabla\times\vec{V}(\boldsymbol{r}))$

define $\mathrm{U(r)} = \nabla\cdot\vec{V}(\boldsymbol{r})$ and $\vec{W}(\mathrm{r}) = \nabla\times\vec{V}(\boldsymbol{r}) \Rightarrow \vec{V}(r) = \nabla U(\boldsymbol{r}) - \nabla\times\vec{W}(\boldsymbol{r})$

(Both U(r), $\vec{W}(\boldsymbol{r})$ have to go to zero faster than $1/r^2$ as $r \to \infty$)

: boundary condition

$$\boldsymbol{V}(r) = \nabla U(\boldsymbol{r}) - \nabla \times \boldsymbol{W}(\boldsymbol{r})$$

The divergence and curl of a vector field uniquely define a vector field.

- So any vector can be written as a sum of a Divergence less field and a Curl less field.