Vector Analysis -2

Electromagnetic Theory
PHYS 401

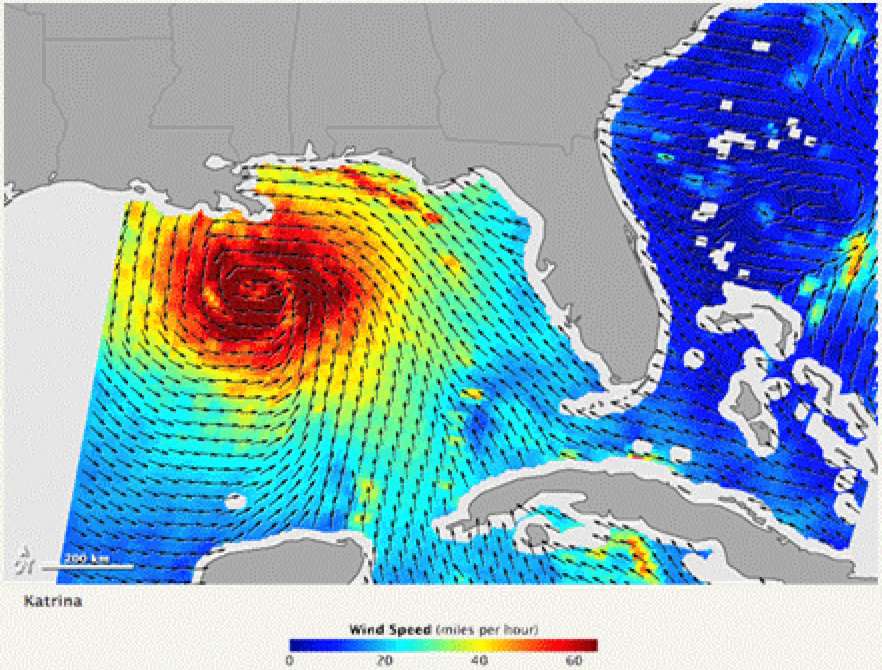
Fall 2018

Vector Calculus

Fields:

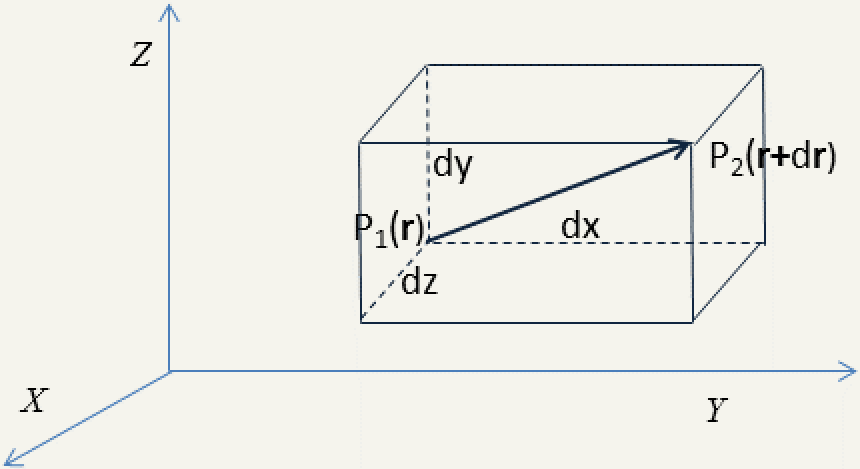
A field can be defined as a function that specifies a value for a particular quantity everywhere in a region. It could be a scalar, vector or other type of field.

- Air temperature in a room: every location has a specific temperature so temperature can be considered as scalar field.
- Wind speed: Speed of air in the atmosphere is another example, since wind speed has a direction (velocity), it is a vector field.



Wind speed at sea level during hurricane Katrina

Gradient of a scalar Field



- Suppose a certain scalar field (i.e. temperature) given by $T(x,y,z)$

    temperature at the point P_1 is $T(\mathbf{r}) = T(x,y,z)$

    temperature at the point P_2 is $T(\mathbf{r}+d\mathbf{r})=T(x+dx,y+dy,z+dz)$

The displacement from P1 to P2 is the displacement vector $d\boldsymbol{r}$ with components (dx,dy,dz). $d\boldsymbol{r} = dx\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}} + dz\hat{\boldsymbol{z}}$

Temperature difference between P1,P2: $dT = T(\mathbf{r}+\mathbf{dr})-T(\boldsymbol{r})$

$$dT=T(x+dx,y+dy,z+dz) - T(x,y,z)$$

$$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

But $\quad dx = \hat{x}\cdot d\mathbf{r} \quad dy = \hat{y}\cdot d\mathbf{r} \quad dz = \hat{z}\cdot d\mathbf{r}$

so $$dT = \frac{\partial T}{\partial x}dx + \frac{\partial T}{\partial y}dy + \frac{\partial T}{\partial z}dz$$

$$= \frac{\partial T}{\partial x}\hat{\mathbf{x}}\cdot d\mathbf{r} + \frac{\partial T}{\partial y}\hat{\mathbf{y}}\cdot d\mathbf{r} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\cdot d\mathbf{r} = \left(\frac{\partial T}{\partial x}\hat{\mathbf{x}} + \frac{\partial T}{\partial y}\hat{\mathbf{y}} + \frac{\partial T}{\partial z}\hat{\mathbf{z}}\right)\cdot d\mathbf{r}$$

So the change in temperature dT is given as the projection of a 'change of temperature vector' (inside square brackets) corresponds to the displacement d**r** .

This vector is called the **Gradient** of the scalar T,

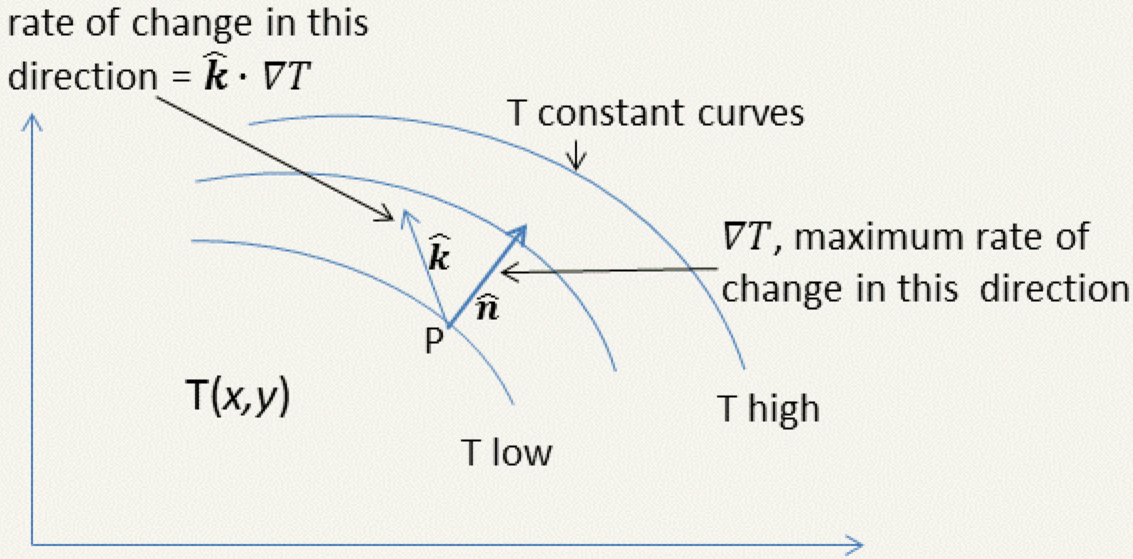
written as Grad T or ∇T.

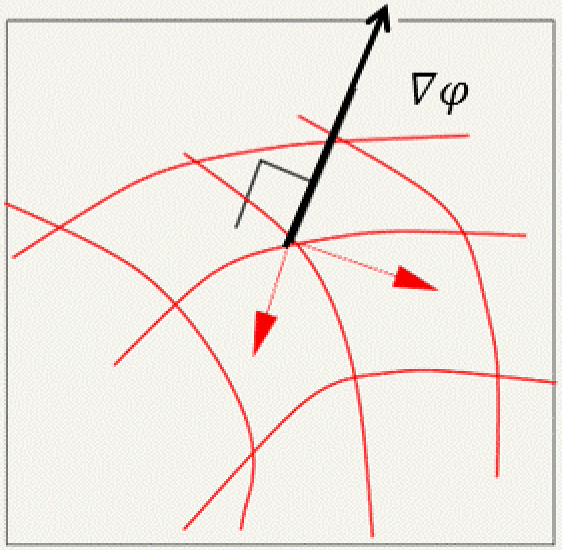
It is a generalization of one the dimensional differential operator

$$\frac{df}{dx} \equiv \lim_{\Delta x \to 0}\left(\frac{1}{\Delta x}\left[f(x+\Delta x) - f(x)\right]\right) \to df = \frac{df}{dx}dx$$

to 3 (or higher) dimensions: $df = \lim_{\Delta\mathbf{r}\to 0}\left(\left[f(\mathbf{r}+\Delta\mathbf{r}) - f(\mathbf{r})\right]\right) = \nabla f\cdot d\mathbf{r}$

Gradient operator





A surface of constant φ (a level surface), $\nabla\varphi$ is normal to it.

- Gradient is the 'slope' of a scalar field at a point. It gives the direction and magnitude of the greatest rate of change of the field.
- Rate of change in any other direction is given by the projection of gradient in that direction
- Rate of change in a given direction = $\hat{\boldsymbol{k}} \cdot \nabla T$,

  $\hat{\boldsymbol{k}}$ unit vector in that direction

‘Del’ Operator

$$\nabla \mathrm{T} = \hat{\mathbf{x}}\frac{\partial \mathrm{T}}{\partial \mathrm{x}} + \hat{\mathbf{y}}\frac{\partial \mathrm{T}}{\partial \mathrm{y}} + \hat{\mathbf{z}}\frac{\partial \mathrm{T}}{\partial \mathrm{z}}$$

can be considered as the operator $\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial \mathrm{x}} + \hat{\mathbf{y}}\frac{\partial}{\partial \mathrm{y}} + \hat{\mathbf{z}}\frac{\partial}{\partial \mathrm{z}}$ acting on the scalar field T.

It is called the ‘del’ operator when grad T is written as ∇T it represent this process.

Example:

- Find the directional derivative of $f(x,y,z)= x^2 +y^2 +z^2$ along the direction $3\hat{x} + 2\hat{y} - \hat{z}$ and evaluate it at the point (2,1, 2).

$$\nabla f = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\left(x^2 + y^2 + z^2\right) = 2x\hat{x} + 2y\hat{y} + 2z\hat{z}$$

Lets denote the given direction as **a**

The unit vector in the direction of **a**= $\hat{a} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3\hat{x}+2\hat{y}-\hat{z}}{\sqrt{3^2+2^2+(-1)^2}} = \frac{3\hat{x}+2\hat{y}-\hat{z}}{\sqrt{14}}$

directional derivative in the direction $3\hat{x} + 2\hat{y} - \hat{z} = \hat{a} \cdot \nabla f$

$$= \frac{3\hat{x}+2\hat{y}-\hat{z}}{\sqrt{14}} \cdot (2x\hat{x} + 2y\hat{y} + 2z\hat{z}) = \frac{6x+4y-2z}{\sqrt{14}}$$

at the point (2,1,2) = $\frac{6\cdot 2+4\cdot 1-2\cdot 2}{\sqrt{14}} = \frac{12}{\sqrt{14}}$

Find the gradient of $r = \sqrt{x^2 + y^2 + z^2}$ (the magnitude of the position vector).

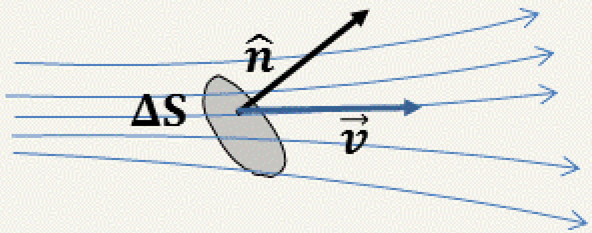
Problem 1.13 Let $\boldsymbol{\mathscr{r}}$ be the separation vector from a fixed point (x', y', z') to the point (x, y, z), and let $\mathscr{r}$ be its length. Show that

(a) $\nabla(\mathscr{r}^2) = 2\boldsymbol{\mathscr{r}}$.

(b) $\nabla(1/\mathscr{r}) = -\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2$.

(c) What is the *general* formula for $\nabla(\mathscr{r}^n)$?

Divergence of a Vector Field



Flux (amount of flow) through area $\Delta S = \Delta S v cos\theta$

$= \Delta\vec{S} \cdot \vec{v}$

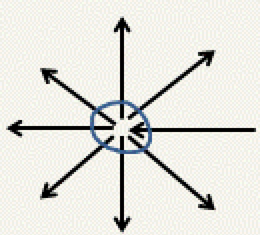
vector area (directed area)

- The **divergence** *of* a vector field at a given point is the net outward flux per unit volume. It is a scalar quantity.

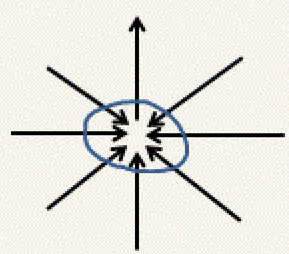
$$\text{div } \vec{A} = \nabla \cdot \vec{A} = \frac{\text{net outward flux of } \vec{\mathbf{A}} \text{ over an infinitesimal volume}}{\text{volume}}$$

$$= \lim_{\Delta V \to 0} \frac{\oint_{\Delta S} \vec{A} \cdot d\vec{S}}{\Delta V}$$ flow of A integrated over the surface

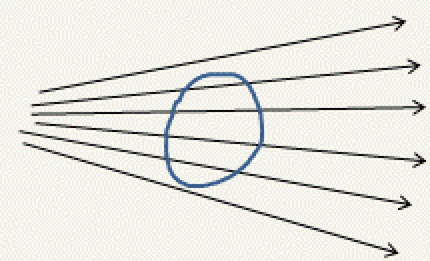
It represents the amount of field sources at each point.



Net outward flow, positive divergence

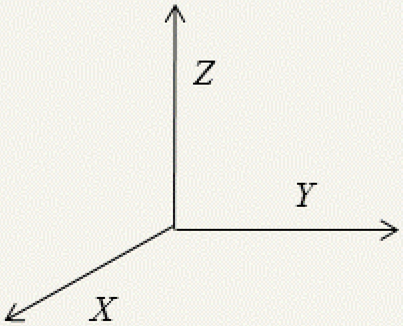


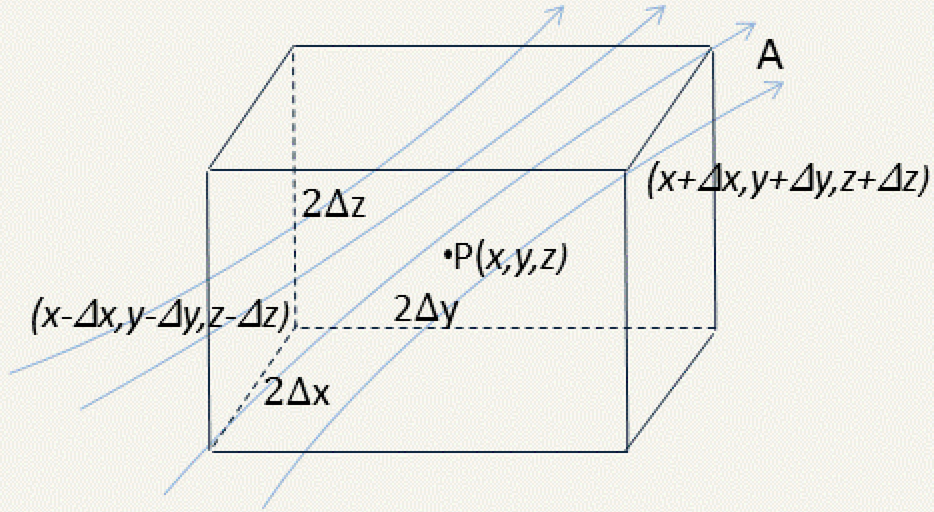
Net inward flow, positive divergence



No net outward flow zero divergence

Divergence in Cartesian coordinates:

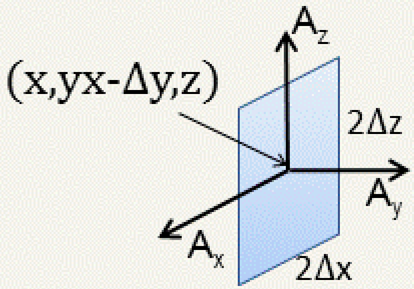




- Suppose a vector filed A =(A_x,A_y,A_z) at the the point (x,y,z).

Lets calculate the net outward flus of this field over a small rectangular box of size (2Δx,2Δy,2Δz) centered at (x,y,z)

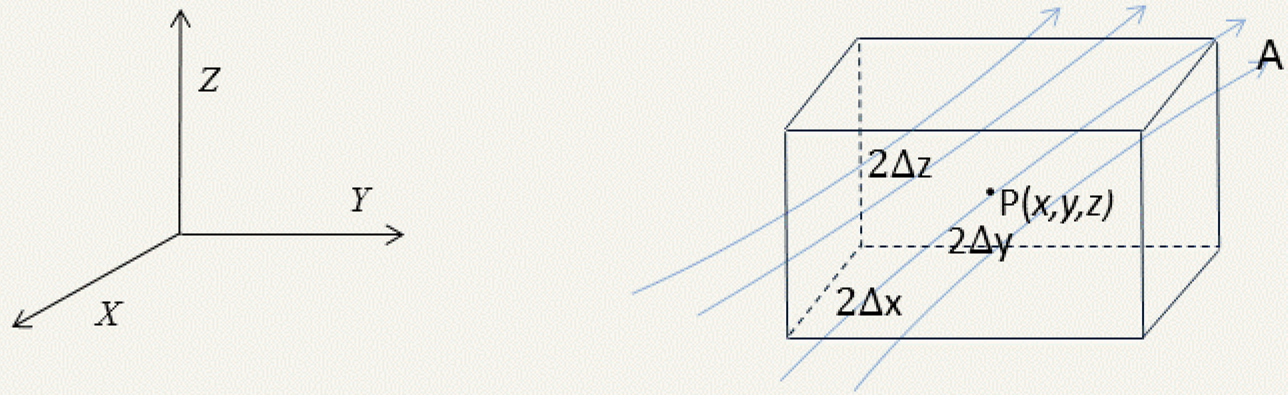
On the left face of the box outward flux is



$$\simeq -\underbrace{A_y(x, y-\Delta y, z)}_{\text{field normal to face at the center of face}}.\underbrace{2\Delta z.2\Delta x}_{\text{area}} = -\left(A_y(x,y,z) - \frac{\partial A_y}{\partial y}\Delta y\right).4\Delta z\Delta x$$

On the right face of the box outward flux

$$\simeq A_y(x, y+\Delta y, z).2\Delta z.2\Delta x = \left(A_y(x,y,z) + \frac{\partial A_y}{\partial y}\Delta x\right).4\Delta z\Delta x$$



Total outward flux from left and right faces

$$= -\left(A_y(x,y,z) - \frac{\partial A_y}{\partial y}\Delta y \right).4\Delta z\Delta x + \left(A_y(x,y,z) + \frac{\partial A_y}{\partial y}\Delta y \right).4\Delta z\Delta x = 8\frac{\partial A_y}{\partial y}\Delta x\Delta y\Delta z$$

Similarly

Total outward flux from front and back faces $= 8\frac{\partial A_x}{\partial x}\Delta x\Delta y\Delta z$

Total outward flux from top and bottom faces $= 8\frac{\partial A_z}{\partial z}\Delta x\Delta y\Delta z$

Total outward flux $= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) \cdot 8\Delta x\Delta y\Delta z = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)\Delta V$

Divergence $= \lim_{\Delta V \to 0} \frac{\oint_{\Delta S} A \cdot dS}{\Delta V} = \boxed{\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}}$

Divergence of the vector field A $= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

$$= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\cdot\left(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}\right)$$

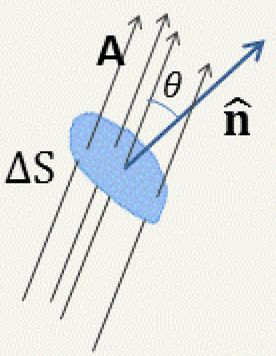
$$= \nabla\cdot\mathbf{A}$$

<u>Divergence Theorem:</u>

The total outward flux of a vector field **A** at the closed surface **S** is the same as volume integral of divergence of **A**.

$$\oint_S \mathbf{A}\cdot d\mathbf{S} = \int_v \nabla\cdot\mathbf{A}\, dV$$

Here $\oint_S \mathbf{A}\cdot d\mathbf{S}$ is the surface integral of **A** over the surface



flux through the surface element of area ΔS $=$ area $\cdot$ A field nornal to it

$$= \Delta S|A|\cos\theta = \Delta S\hat{n}\cdot\mathbf{A}$$

$$= \Delta\mathbf{S}\cdot\mathbf{A}$$

Δ***S***: directed area element (direction of area is the normal direction)

Problem 1.15 Calculate the divergence of the following vector functions:

(a) $\mathbf{v}_a = x^2\,\hat{\mathbf{x}} + 3xz^2\,\hat{\mathbf{y}} - 2xz\,\hat{\mathbf{z}}$.

(b) $\mathbf{v}_b = xy\,\hat{\mathbf{x}} + 2yz\,\hat{\mathbf{y}} + 3zx\,\hat{\mathbf{z}}$.

(c) $\mathbf{v}_c = y^2\,\hat{\mathbf{x}} + (2xy + z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}$.

Problem 1.16 Sketch the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2},$$

and compute its divergence. The answer may surprise you…can you explain it?

The Curl of a Vector Field

Since the *del* operator $\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$ is a vector, it is possible to take the vector product of it with a vector field.

The vector field produced by this operation is called the *curl* of the vector field.

$$\nabla \times \mathbf{A} = \text{curl }\mathbf{A} = \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right) \times \left(A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}} + A_z\hat{\mathbf{z}}\right)$$

$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

The Curl of a Vector Field

curl of a vector field describes the infinitesimal rotation ulation) of the vector field.

ssible to

od measure of the circulation is the line integral of the field nd a closed curve at a given point. If the vector field is tional' it contributes to the integral.

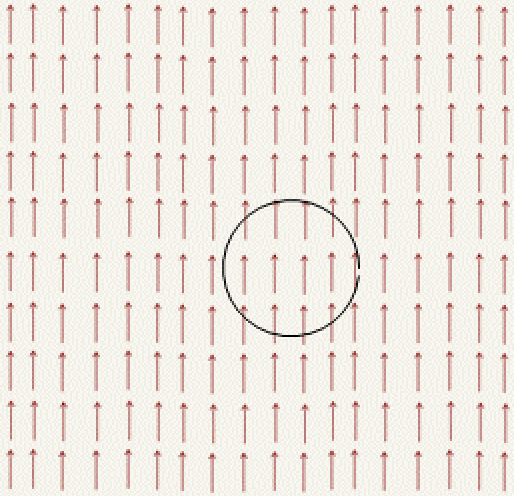
' of the

The line integral of the field **A** along the closed curve C, $\oint_C \mathbf{A}\cdot\Delta\boldsymbol{l}$ is the integral of component (projection) of the vector field along the curve.

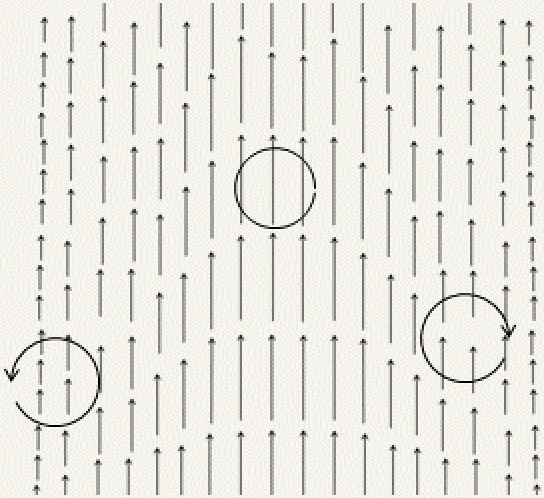
For a small segment of the curve $\Delta\mathbf{l}$:

$$|\Delta\boldsymbol{l}||\mathbf{A}|\cos\theta = \mathbf{A}\cdot\Delta\boldsymbol{l}$$

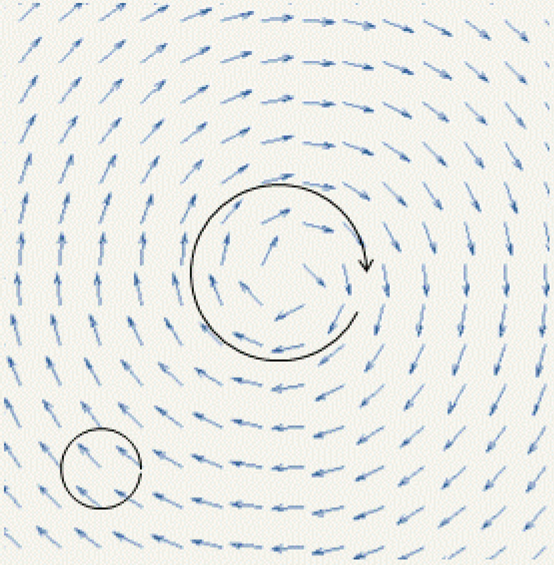
$$\left.\frac{A_x}{y}\right)\hat{\mathbf{z}}$$



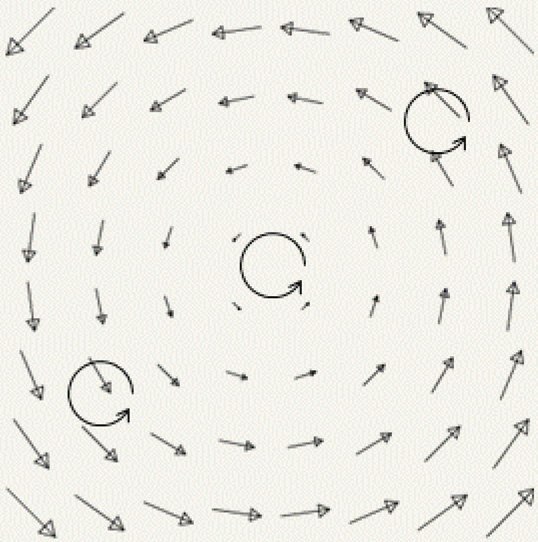
no rotation



rotation exits away
from the center

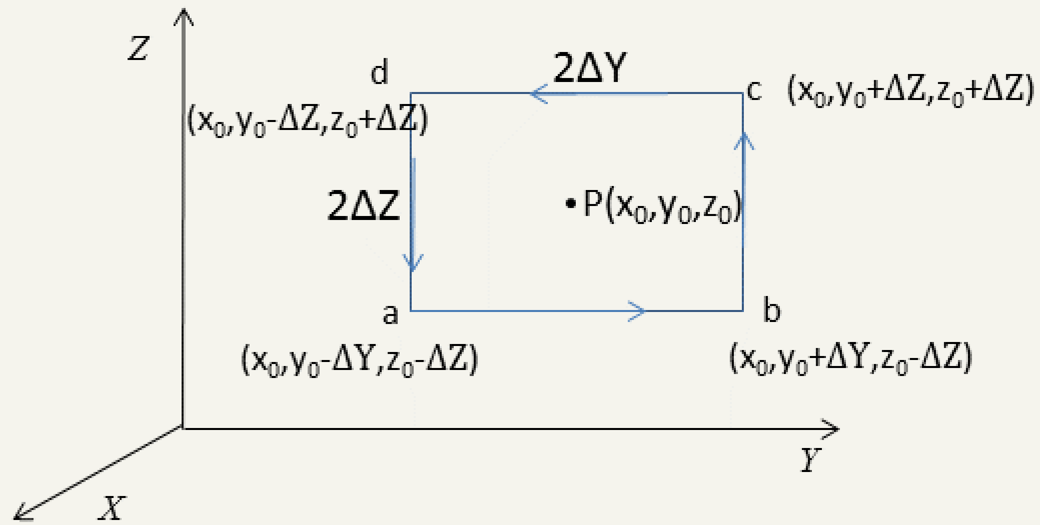


bulk rotation around center,
no local rotation elsewhere



local rotation
everywhere

Watch the video: www.youtube.com/watch?v=vvzTEbp9lrc



To see this lets calculate the circulation of field **A** around a closed rectangular contour *abcd* of size $2\Delta Y\times 2\Delta Z$ around point P(x,y,z), perpendicular to X axis.

$$\oint_{abcd}\mathbf{A}\cdot d\mathbf{l}=\int_{ab}\mathbf{A}\cdot d\mathbf{l}+\int_{bc}\mathbf{A}\cdot d\mathbf{l}+\int_{cd}\mathbf{A}\cdot d\mathbf{l}+\int_{da}\mathbf{A}\cdot d\mathbf{l}$$

$$\int_{ab}\mathbf{A}\cdot d\mathbf{l}=\int_a^b A_y(x_0,y_0,z_0-\Delta Z)dy=\int_a^b\left(A_y(x_0,y_0,z_0)-\Delta Z\frac{\partial A_y}{\partial z}\right)dy=2\Delta Y\left(A_y(x_0,y_0,z_0)-\Delta Z\frac{\partial A_y}{\partial z}\right)$$

evaluated at (x_0,y_0,z_0)

$$\int_{cd}\mathbf{A}\cdot d\mathbf{l}=\int_c^d -A_y(x_0,y_0,z_0-\Delta Z)dy=\int_c^d-\left(A_y(x_0,y_0,z_0)+\Delta Z\frac{\partial A_y}{\partial z}\right)dy=-2\Delta Y\left(A_y(x_0,y_0,z_0)+\Delta Z\frac{\partial A_y}{\partial z}\right)$$

$$\text{so}\ \int_{ab}\mathbf{A}\cdot d\mathbf{l}+\int_{cd}\mathbf{A}\cdot d\mathbf{l}=2\Delta Y\left(A_y(x_0,y_0,z_0)-\Delta Z\frac{\partial A_y}{\partial z}\right)-2\Delta Y\left(A_y(x_0,y_0,z_0)+\Delta Z\frac{\partial A_y}{\partial z}\right)=-4\Delta Z\Delta Y\frac{\partial A_y}{\partial z}$$

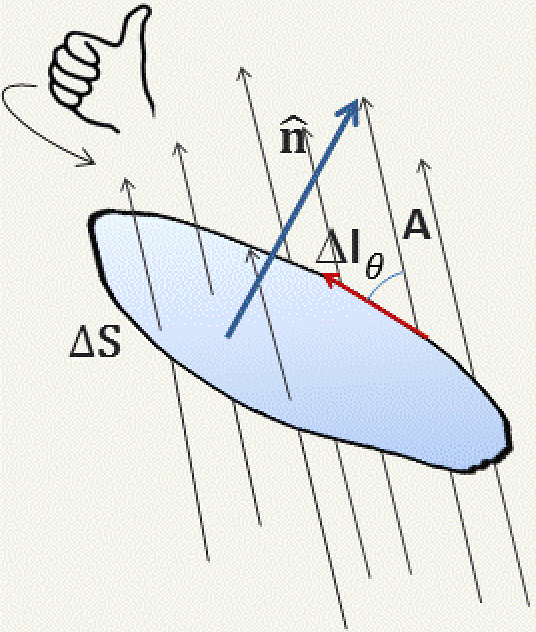
The Curl of a Vector Field

$$\text{similarly} \int_{bd} \mathbf{A}\cdot d\mathbf{l} + \int_{da} \mathbf{A}\cdot d\mathbf{l} = 2 = 4\Delta Z\Delta Y \frac{\partial A_z}{\partial y}$$

$$\oint_{abcd} \mathbf{A}\cdot d\mathbf{l} = \int_{ab} \mathbf{A}\cdot d\mathbf{l} + \int_{bc} \mathbf{A}\cdot d\mathbf{l} + \int_{cd} \mathbf{A}\cdot d\mathbf{l} + \int_{da} \mathbf{A}\cdot d\mathbf{l} = \underbrace{4\Delta Z\Delta Y}_{\text{area of loop}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) = \Delta S\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)$$

$$\frac{1}{\Delta S}\oint_{abcd} \mathbf{A}\cdot d\mathbf{l} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) = \text{x-component of } \nabla\times\mathbf{A}$$

curl of the vector field **A** can be defined as:



$$\text{curl } \mathbf{A} = \nabla\times\mathbf{A} = \lim_{\Delta s\to 0}\frac{1}{\Delta S}\left[\hat{\mathbf{n}}\oint_C \mathbf{A}\cdot d\mathbf{l}\right]_{max}$$

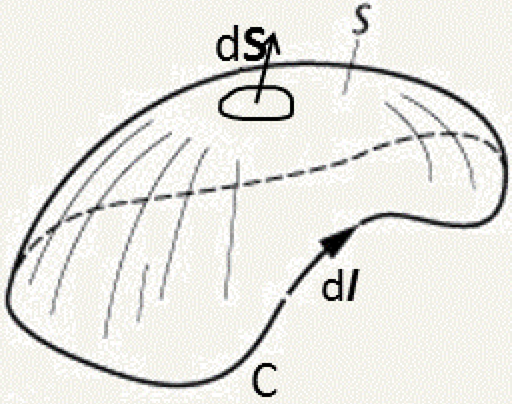
ΔS is the area enclosed by an enclosed curve C oriented such that the integral has maximum value.

$\hat{\mathbf{n}}$ **:** unit vector normal to area ΔS, in the direction of motion of a right handed screw when it is turned in the direction of integral is taken.

Stokes' Theorem

- Stokes' theorem converts the surface integral of the curl of a vector field over an open surface into a line integral of the vector field along the curve bounding the surface.

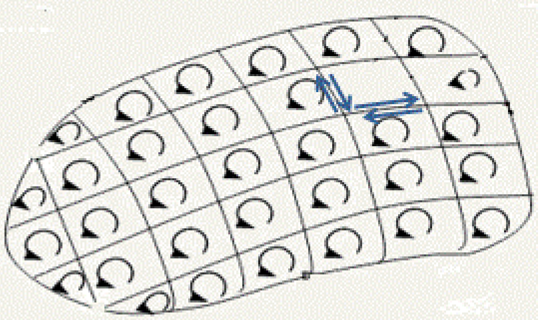
$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_C \mathbf{A} \cdot d\boldsymbol{l}$$



Validity of this can be intuitively seen directly from the definition of curl

$$\nabla \times \mathbf{A} = \lim_{\Delta s \to 0} \frac{1}{\Delta S}\left[\hat{\mathbf{n}} \oint_C \mathbf{A} \cdot \mathbf{dl}\right]_{max} \Rightarrow \Delta S \nabla \times \mathbf{A} \simeq \hat{\mathbf{n}} \oint_C \mathbf{A} \cdot \mathbf{dl}$$

$$\Rightarrow \nabla \times \mathbf{A} \cdot \Delta \mathbf{S} = \oint_{\Delta C} \mathbf{A} \cdot \mathbf{dl}$$



Divide the surface in to a large number of small areas and apply above to each and take the sum

$$\sum_{\text{all areas}} \nabla \times \mathbf{A} \cdot \Delta \mathbf{S} = \sum_{\text{all areas}} \oint_{\Delta C} \mathbf{A} \cdot \mathbf{dl}$$

- the line integrals along the common sides of adjacent areas mutually cancel.
- only those sides in the periphery of the surface contribute to the sum.

∴ In the limit areas are infinitesimal this becomes the Stokes' theorem

Laplacian Operator

The Laplacian operator is the scalar product of the del operator with itself.

$$\nabla^2 = \nabla \cdot \nabla$$

$$= \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right) \cdot \left(\hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

The result is a scalar operator. It can be applied to a scalar or vector field.

For a scalar field ϕ

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

for a vector field $\mathbf{A} = \hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z$

$$\nabla^2 \mathbf{A} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \mathbf{A} = \hat{\mathbf{x}} \nabla^2 A_x + \hat{\mathbf{y}} \nabla^2 A_y + \hat{\mathbf{z}} \nabla^2 A_z$$