Vector Analysis -1

Electromagnetic Theory
PHYS 401

Fall 2018

Vector Analysis

Vector analysis is a mathematical formalism with which EM concepts are most conveniently expressed and best comprehended.

- Many physical quantities are completely describes by their value
  - e.g. temperature, pressure, mass, frequency…
- Such quantities are called scalars, and their values can be given in numbers.
- But many physical quantities have a direction in addition to magnitude
  - e.g.: velocity, force, displacement…
  - To describe such quantities their direction as well as their magnitudes have to be specified. So just a regular number is not inadequate.

Vector: a mathematical object that has a magnitude and a direction.

- Physical quantities which possess a direction as well as a magnitude are represented by vectors

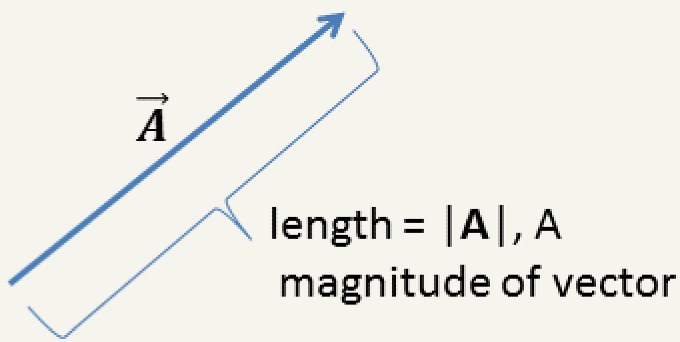
Vector Notation: A vector is usually written as a bold face letter (like **A)** or with an little arrow or line above it $\vec{A}$, $\bar{A}$

The magnitude of a vector **A** is written as |**A**| or A

Vectors

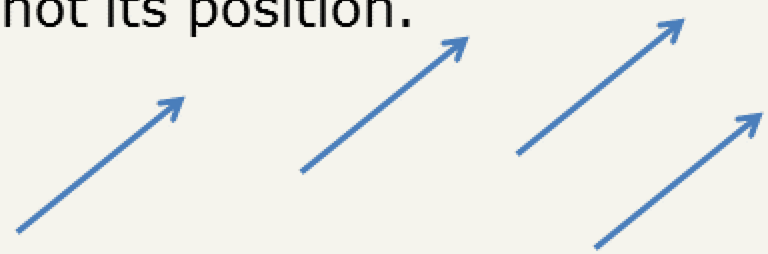
- Often a vector is graphically represented by a directed line segment (an arrow):
  - Whose length represents the magnitude and its orientation in the direction of the vector.

direction: direction of vector



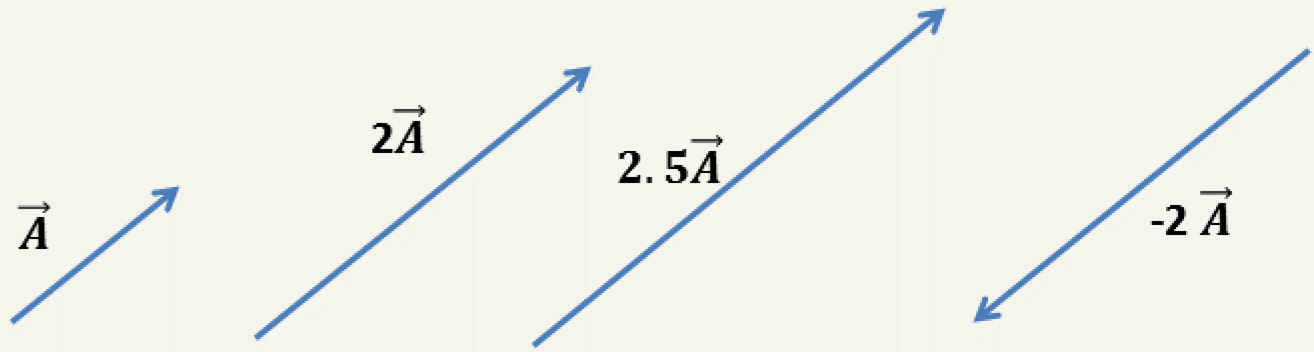
NB.: This is just a geometrical representation, vectors are not arrows, they are abstract mathematical objects.

In representing a vector graphically, it is defined only by its magnitude and direction, not its position.



All these represent the same vector

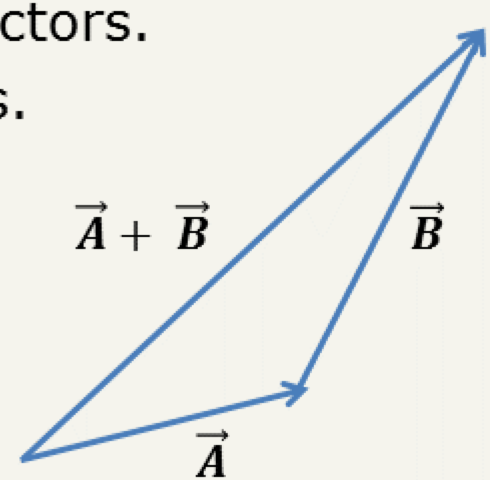
- When a vector is multiplied by a positive number, it multiplies its magnitude, its direction stays the same.
- Multiplying by a negative number flips the direction of a vector



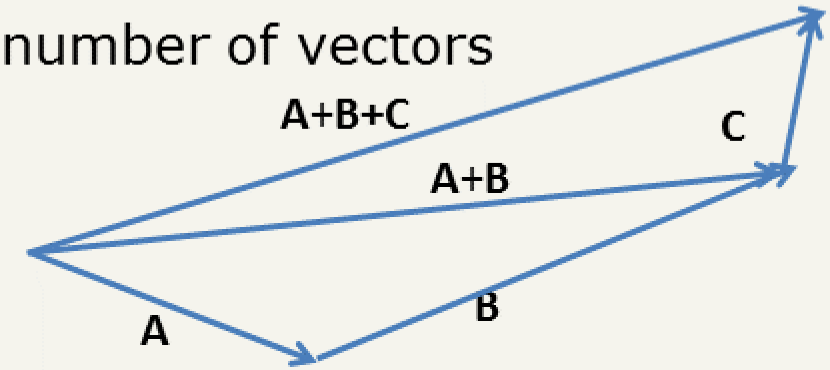
- Sum of two vectors is the single equivalent vector which has same effect as application of the two vectors.
- e.g. consider adding two displacements.

Graphically, vectors are added using the triangular rule

(also called "head-to-tail" rule; Move $\vec{B}$ so that the head of $\vec{A}$ touches the tail of $\vec{B}$)



- This can be repeated to add any number of vectors



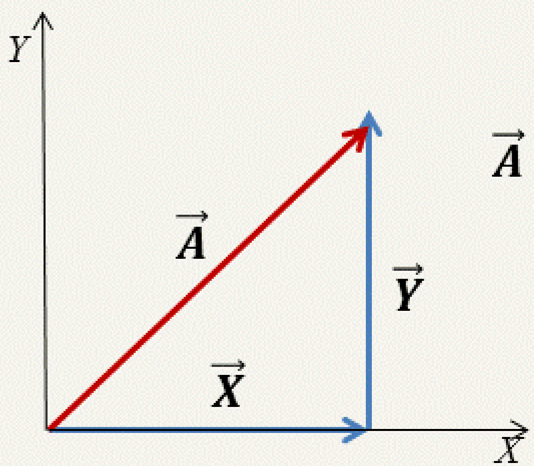
given any three vectors **A, B**, and **C** vector addition obey following properties:

	Addition	Multiplication by a scaler
Commutative	$\overline{A}+\overline{B}=\overline{B}+\overline{A}$	$k\overline{A}=\overline{A}k$
Associative	$\overline{A}+(\overline{B}+\overline{C})=(\overline{A}+\overline{B})+\overline{C}$	$k(l\overline{A})=(kl)\overline{A}$
Distributive	$k(\overline{A}+\overline{B})=k\overline{A}+k\overline{B}$	

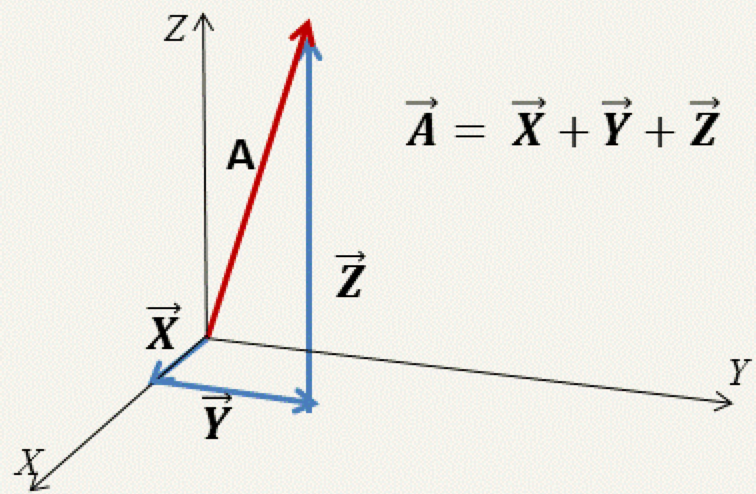
where k and l are scalars

Vector Decomposition

- Just like vectors can be added to form a single vector, any given vector can be written as a combination of other vectors.
- This is called vector decomposition.
- One particularly useful decomposition is, decomposing a vector as a sum of vectors parallel to coordinate axes.



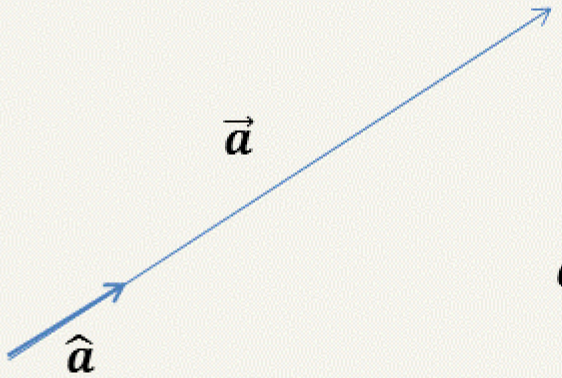
$$\vec{A} = \vec{X} + \vec{Y}$$



$$\vec{A} = \vec{X} + \vec{Y} + \vec{Z}$$

- Those are called components of the vector along coordinate axes

- Unit vectors: A vector whose magnitude is 1.
  - Usually unit vectors are written with a hat (like $\hat{\boldsymbol{a}}$, $|\hat{\boldsymbol{a}}|=1$)
  - Any vector can be written as a product of its magnitude times the unit vector in that direction.



$\vec{\boldsymbol{a}} = |\boldsymbol{a}|\ \hat{\boldsymbol{a}} \quad \Rightarrow \quad \hat{\boldsymbol{a}} = \dfrac{\vec{\boldsymbol{a}}}{|\boldsymbol{a}|}$

$\hat{\boldsymbol{a}}$: the unit vector in the direction of the vector $\boldsymbol{a}$

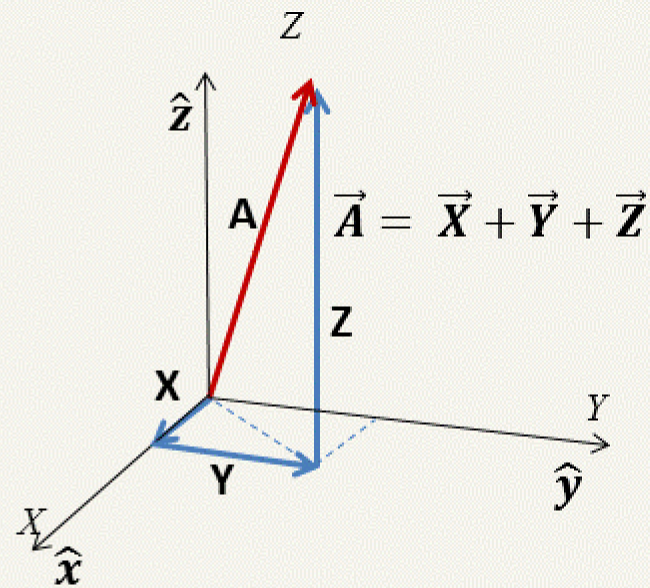
Suppose $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ are unit vectors along X,Y,Z coordinate directions,

Then $\vec{\boldsymbol{X}} = |\boldsymbol{X}|\ \hat{\boldsymbol{a}}$, let $|\mathbf{X}| = x$ then $\vec{X} = \mathrm{x}\hat{\boldsymbol{x}}$

Similarly $\vec{Y} = \mathrm{y}\hat{\boldsymbol{y}}$, $\mathbf{Z} = z\,\hat{\boldsymbol{z}}$

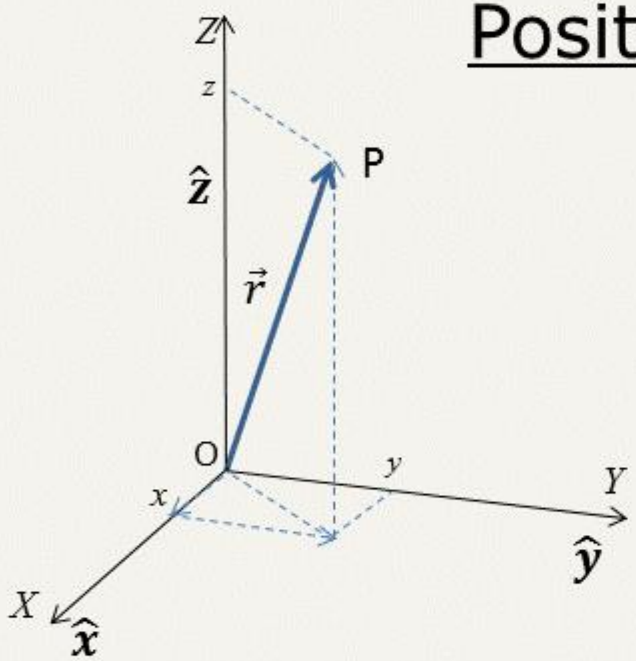
So $\vec{\boldsymbol{A}} = \vec{\boldsymbol{X}} + \vec{\boldsymbol{Y}} + \vec{\boldsymbol{Z}} = x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}} + z\,\hat{\boldsymbol{z}}$

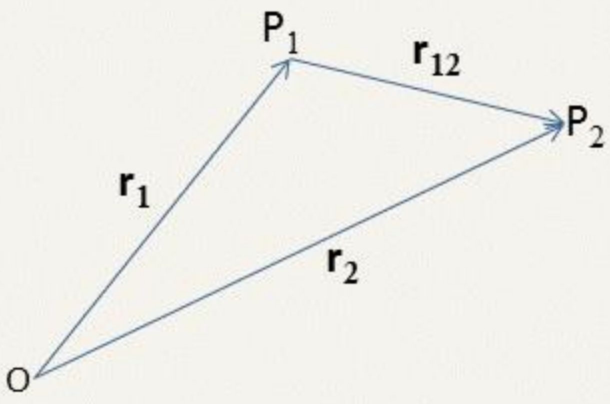
Often it is just written as a coordinate triplet $\vec{\boldsymbol{A}} = (x,y,z)$ leaving the sum over unit vectors to be understood.



Magnitude of the vector $\vec{\boldsymbol{A}} = \sqrt{x^2 + y^2 + z^2}$

Position Vector (Radius Vector)





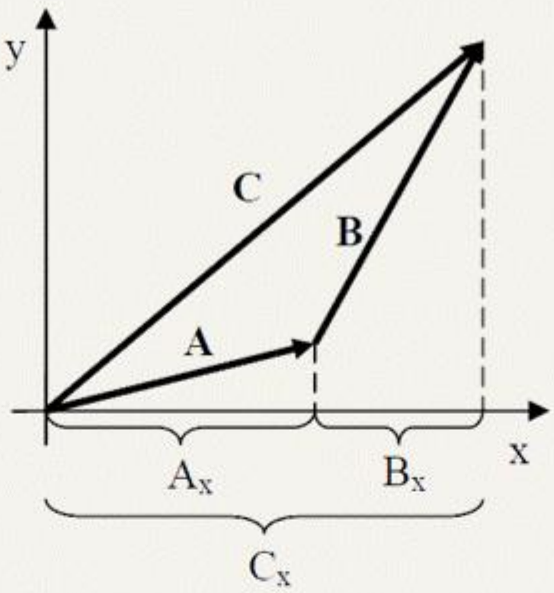
The position vector of point P is defined as the directed distance from the origin O to P; that is,

$$\vec{\mathbf{r}} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$$

- The unit vector in the direction of $\vec{r}$ is

$$\hat{\boldsymbol{r}} = \frac{\vec{r}}{|\boldsymbol{r}|} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}}$$

- If there are two points with position vectors $\mathbf{r_1}$ and $\mathbf{r_2}$, the ***distance vector*** between them $\vec{\mathbf{r}}_{12} = \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1$



- Components of sum of vectors is sum of components

e.g. Let $\mathbf{A} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\,\hat{\mathbf{z}}$ and $\mathbf{B} = B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\,\hat{\mathbf{z}}$

Then A+B = $(A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\,\hat{\boldsymbol{k}}\,) + (B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\,\hat{\mathbf{z}}\,)$

$= A_x\hat{\boldsymbol{x}} + B_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + B_y\hat{\boldsymbol{y}} + A_z\,\hat{\mathbf{z}} + B_z\,\hat{\mathbf{z}}$

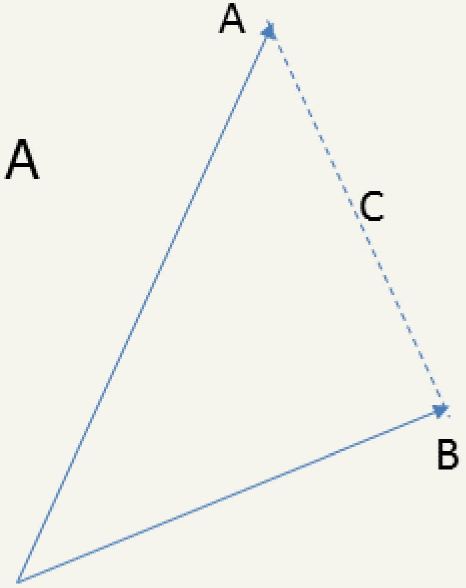
$= (A_x + B_x)\hat{\boldsymbol{x}} + (A_y + B_y)\hat{\boldsymbol{y}} + (A_z + B_z\,)\,\hat{\mathbf{z}}$

Example:

Position vectors of points A, B are

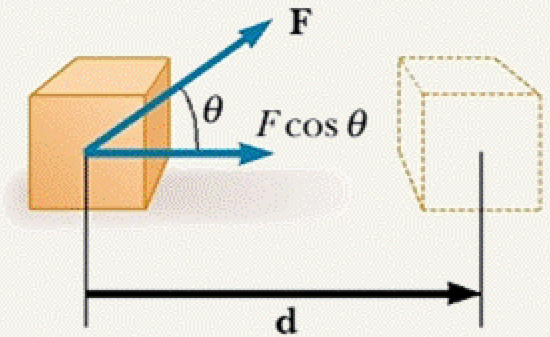
$2\hat{\boldsymbol{x}} + 4\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}$ and $8\hat{\boldsymbol{x}} + 12\hat{\boldsymbol{y}} + 13\,\hat{\boldsymbol{z}}$ respectively.

What is the position vector of the point C which is located half way on the straight in between A and B.

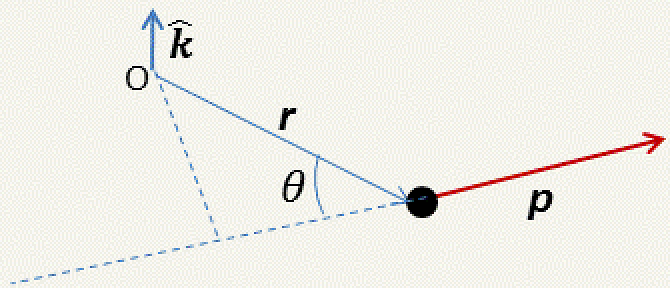


Vector Products

- Vector quantities are often combined to form a new quantities. the result could be a scale or a vector.



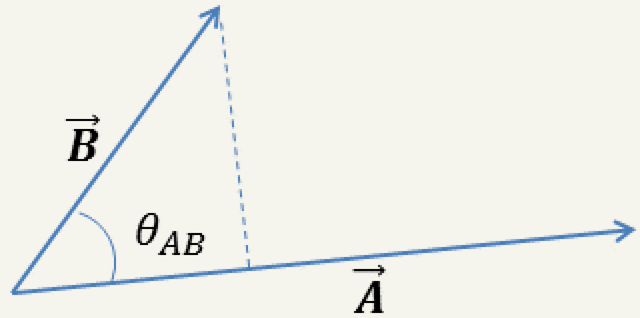
$$W = dF\cos\theta$$



$$\boldsymbol{L} = \hat{\boldsymbol{k}}\, rp\sin\theta$$

  - e.g. Work done by a force is the product of displacement times the force in the direction of displacement.
    - Both force and displacement are vector quantities, and work is a scalar.
  - Angular momentum is the product of momentum and distance perpendicular to momentum to origin (axis of rotation).
    - Momentum is a vector, since distance is taken in a specified direction it also becomes a vector. Resulting angular momentum is also a vector.
- So two types of vector products are defined, scalar and vector.

The Scalar Product (Dot Product)



$\vec{A}$

The scalar product (also called of the two vectors $\vec{A}$ and $\vec{B}$ is defined as the product of the magnitude of $\vec{A}$ and the projection of $\vec{B}$ onto $\vec{A}$ (or vice versa):

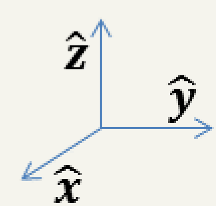
$$\vec{A} \cdot \vec{B} = \mathrm{AB} \cos \theta_{AB}$$

where θ_{AB} is the angle between **A** and **B.**

dot product is commutative : $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$

and distributive : $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$

- If two vectors are perpendicular to each other their scaler product is zero ($\cos 90^\circ = 0$) .
- Therefore for unit vectors $\hat{x}, \hat{y}, \hat{z}$ along coordinate axes



$$\hat{x} \cdot \hat{y} = 0;\ \hat{x} \cdot \hat{z} = 0;\ \hat{y} \cdot \hat{z} = 0$$

and $\hat{x} \cdot \hat{x} = 1;\ \hat{y} \cdot \hat{y} = 1;\ \hat{z} \cdot \hat{z} = 1$

if **A** = $A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\mathbf{z}}$ and **B** = $B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\mathbf{z}}$

$$\mathbf{A}\cdot\mathbf{B} = (A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\mathbf{z}})\cdot(B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\mathbf{z}})$$

using above properties $= A_xB_x + A_yB_y + A_zB_z$

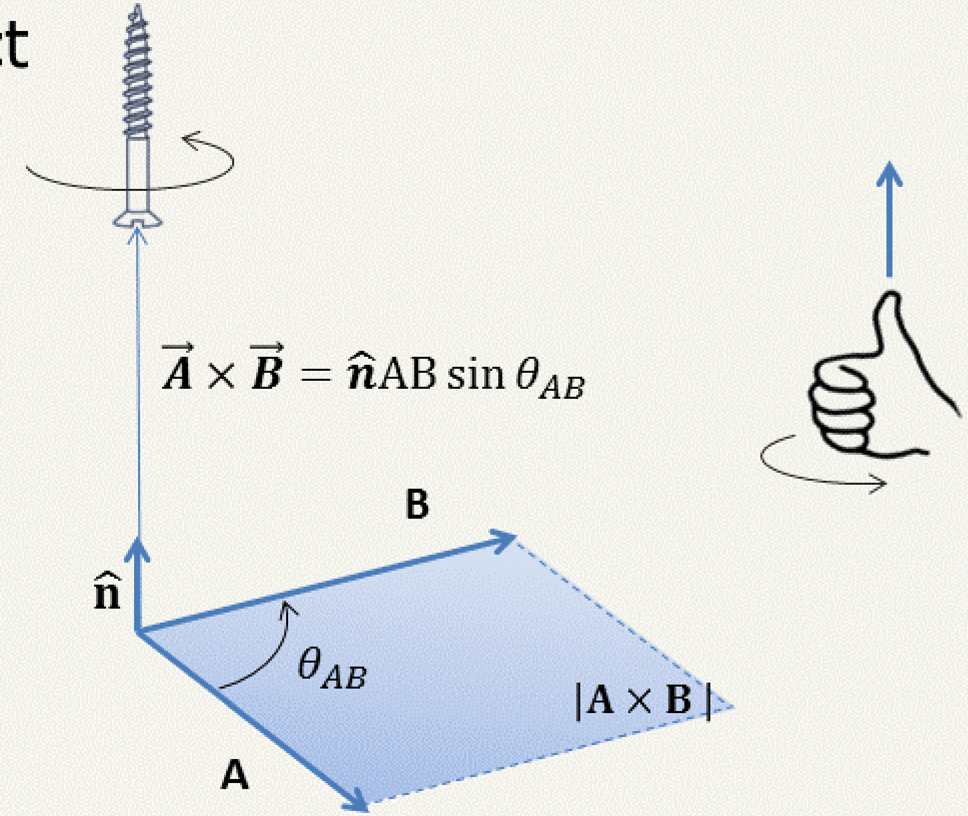
$$\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{A}} = A_xA_x + A_yA_y + A_zA_z = A_x{}^2 + A_y{}^2 + A_z{}^2 = |\mathbf{A}|^2$$

Since $\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}} = \mathrm{AB}\cos\theta_{AB}$

$\cos\theta_{AB} = \dfrac{\vec{\boldsymbol{A}}\cdot\vec{\boldsymbol{B}}}{|\mathbf{A}||\mathbf{B}|}$ useful in finding angle between two vectors.

In general the component of a vector $\vec{\boldsymbol{A}}$ in the direction of vector $\vec{\boldsymbol{C}}$ is given by $\vec{\boldsymbol{A}}\cdot\hat{\mathbf{C}}$

The Vector Product (Cross Product)



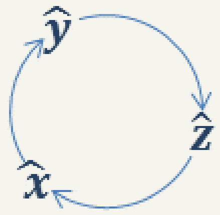
- The vector product of two vectors, $\vec{A}$ and $\vec{B}$ is a **vector**,
  - its magnitude is equal to the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the sine of the angle between them
  - its direction is perpendicular to the plane containing $\vec{A}$ and $\vec{B}$, in the direction of advance of a right handed screw when it is turned from **A** to **B.**

$$\vec{A} \times \vec{B} = \hat{n}\text{AB}\sin\theta_{AB}$$

$|\vec{A} \times \vec{B}|$ = area of the parallelogram determined by **A** and **B**

- The vector product has following properties:
  - It is not commutative: $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$
  - It is not associative: $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$
  - It is distributive: $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$
  - $\mathbf{A} \times \mathbf{A} = \mathbf{0} \quad (\sin\theta = 0)$

- For unit vectors $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\mathbf{z}}$ along coordinate axes



$$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{x}} = \mathbf{0}; \quad \hat{\boldsymbol{y}} \times \hat{\boldsymbol{y}} = \mathbf{0}; \quad \hat{\mathbf{z}} \times \hat{\mathbf{z}} = \mathbf{0}$$

$$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = \hat{\mathbf{z}}; \quad \hat{\boldsymbol{y}} \times \hat{\mathbf{z}} = \hat{\boldsymbol{x}}; \quad \hat{\mathbf{z}} \times \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}$$

$\hat{\boldsymbol{y}} \times \hat{\boldsymbol{x}} = -\hat{\mathbf{z}}$ ... and so on

In tems of components

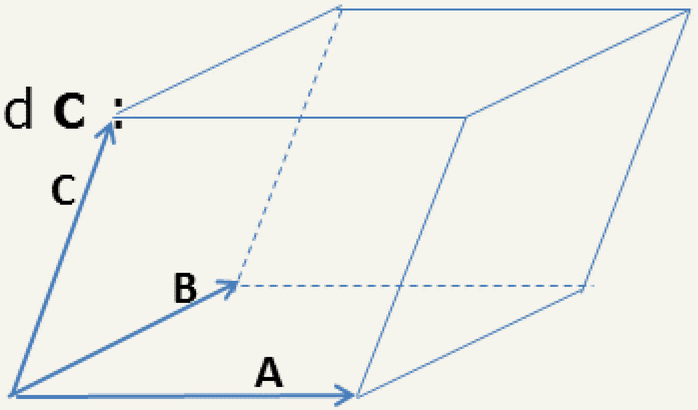
$$\mathbf{A} = A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\mathbf{z}} \text{ and } \mathbf{B} = B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\mathbf{z}}$$

$$\begin{aligned}\mathbf{A} \times \mathbf{B} &= (A_x\hat{\boldsymbol{x}} + A_y\hat{\boldsymbol{y}} + A_z\hat{\mathbf{z}}) \times \left(B_x\hat{\boldsymbol{x}} + B_y\hat{\boldsymbol{y}} + B_z\hat{\mathbf{z}}\right) \\ &= (A_yB_z - A_zB_y)\,\hat{\boldsymbol{x}} + (A_zB_x - A_xB_z)\,\hat{\boldsymbol{y}} + (A_xB_y - A_yB_z)\,\hat{\mathbf{z}} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}\end{aligned}$$

Triple vector products

- Since result of the vector product between two vectors gives a vector it can it can be multipied with another vector.

- The scalar triple product between **A** ,**B** and **C** :



$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=-\mathbf{A}\cdot(\mathbf{C}\times\mathbf{B})$$

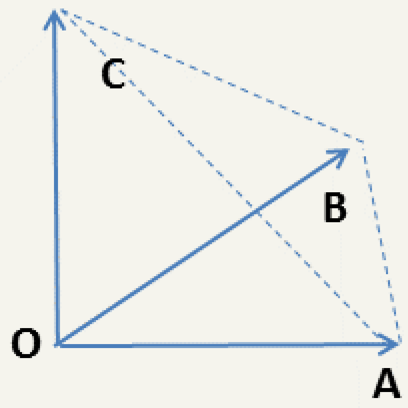
  – It is equal to the volume of the parallelepiped spanned by **A**, **B**, and **C**

In terms of components $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$

- The vector triple product between **A** ,**B** and **C** :

$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\mathbf{B}(\mathbf{A}\cdot\mathbf{C})-\mathbf{C}(\mathbf{A}\cdot\mathbf{B})$ ("bac-cab" rule)

It is not associative: $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\neq(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$



Example:

Position vectors of point A, B, C are $4\hat{\boldsymbol{x}}$, $3\hat{\boldsymbol{y}}$ and , $5\hat{\boldsymbol{z}}$ with respect to a Cartesian coordinate system with origin O.

What is the unit vector normal to the triangle ABC?

What is the area of the triangle ABC?

What is the angle CAB?

What is the volume of the pyramid OABC.