Special Relativity I: Spacetime

Basic Definitions and Postulates

• *Event, spacetime manifold:* Spacetime is the set of all possible locations of physical phenomena, in space and time. In ordinary approaches it is modeled as a 4-dimensional manifold M, a set of elements called events with suitable sets of coordinate systems used to identify events.

• *Inertial coordinates:* An inertial coordinate system $\{x^\mu\} = \{(t, \mathbf{x})\}$ on M is one based on an inertial observer. Conventionally, $\mu = 0, 1, 2, 3$; $x^0 = t$ is time measured by a clock carried by the inertial observer. The location (worldline) of that observer is assumed to be at $\mathbf{x} = 0$. Here we will only consider explicitly Cartesian coordinates. Coordinate system of other types (spherical, cylindrical, etc) are related to Cartesian ones (with the same t axis) by the usual coordinate transformations.

• *Postulate 1:* (Relativity) The laws of physics are the same in all inertial reference frames. This principle was in effect used in prerelativity physics, but it gained prominence only with special relativity.

• *Postulate 2:* (Speed of light) The speed of light has the same value in all reference frames. This fact is highly non-intuitive, but it follows from the results of the Michelson-Morley experiment.

• *Causality:* Because the speed of light is independent of the source, each event $p \in M$ has an absolute past (the set of events that can influence p) and a future (the set of events that p can influence). The boundaries of those sets are the light cone of p, representing events that can be reached moving towards/from p at the speed of light. Knowing the light cones of all $p \in M$ is equivalent to knowing the causal structure of M.

• *Worldline:* A 1D line in M representing the set of locations of a particle in time. Because particles cannot move faster than light, each point along a worldline must be in the past or future of every other one.

• *Coordinates, space, simultaneity:* Coordinate systems (when not considering gravity and curved spacetime) ordinarily consist of one timelike coordinate t whose value represents the reading of a clock at each location, and three spacelike coordinates $\mathbf{x}$. Space is the set of all events in M with the same value of t; the geometry of such a surface is the usual Euclidean one, and different events in it are called simultaneous.

Lorentz Trasformations

• *Poincaré and Lorentz trasformations:* Poincaré transformations are the most general linear maps $\Lambda : M \to M$ that take inertial coordinate systems to other inertial coordinate systems, consistently with the speed of light postulate. The simplest ones are translations $x^\mu \mapsto x^\mu + a^\mu$, where the a^μ are constants. The remaining ones, rotations and boosts, are Lorentz trasformations. Rotations can be represented by matrices R such that $\mathbf{x} \mapsto \mathbf{x}' = R\,\mathbf{x}$, as in Euclidean geometry. For example, for a rotation by an angle θ around the z axis,

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} .$$

Boosts are transformations between frames set up by observers with non-zero relative velocities. For example, if observer O' is moving with velocity $\mathbf{v} = v\,\mathbf{i}$ with respect to observer O, then $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, where

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} , \qquad \gamma := (1-\beta^2)^{-1/2} , \qquad \beta := v/c .$$

• *Invariant interval:* For any $\Delta x^\mu = x'^\mu - x^\mu$, all Poincaré transformations leave invariant the expression

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 ,$$

which therefore represents a coordinate independent, physically meaningful quantity characterizing the relationship between two events: it is the proper time elapsed $\Delta\tau = \sqrt{-\Delta s^2}$ if $\Delta s^2 < 0$, or the proper length $L_0 = \sqrt{\Delta s^2}$ if $\Delta s^2 > 0$. Pairs of events for which Δs^2 is negative, zero or positive, respectively, are called timelike, null and spacelike related. Note: We will usually assume that in our units the numerical value of the speed of light is 1; for example, we might be measuring times in seconds and distances in light-seconds.

• *The Lorentz group:* In Euclidean geometry the squared length of a vector $\mathbf{x}$ is given by $\|\mathbf{x}\|^2 := \mathbf{x} \cdot \mathbf{x}$, which can also be written in matrix form as $\mathbf{x}^{\mathrm{T}}\, 1\, \mathbf{x}$, and the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ can be obtained from the dot product $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^{\mathrm{T}}\, 1\, \mathbf{y}$, where here "1" is the identity matrix. Since a rotation $\mathbf{x} \mapsto R\,\mathbf{x}$ (under which $\mathbf{x}^{\mathrm{T}} \mapsto \mathbf{x}^{\mathrm{T}} R^{\mathrm{T}}$) does not change lengths or angles, the matrix R must be such that

$$R^{\mathrm{T}} 1\, R = R^{\mathrm{T}} R = 1 \;, \quad \text{i.e.}\;, \quad R^{\mathrm{T}} = R^{-1} \; (R \text{ is an } \textit{orthogonal} \text{ matrix})\;.$$

If we take the determinant of the equality $R^{\mathrm{T}} 1\, R = 1$ ("R preserves the matrix 1") and use $\det R = \det R^{\mathrm{T}}$, we get $(\det R)^2 = 1$. A matrix with $\det R = -1$ would invert the orientation of the axes, so we choose $\det R = 1$ (R is a *special* matrix). Because rotations are 3×3 matrices, we end up with the group SO(3).

Lorentz Trasformations and Minkowski Spacetime

• *The Minkowski metric:* The geometry of special relativity is encoded in an analogous but different property of Lorentz transformations. Since the squared interval can be written as

$$(\Delta s)^2 = \Delta x^\mu\, \eta_{\mu\nu}\, \Delta x^\nu = \Delta x^{\mathrm{T}} \eta\, \Delta x \;,$$

where $\eta_{\mu\nu}$ is the Minkowski metric, given in the usual coordinates by the matrix

$$\eta = \mathrm{diag}(-1, 1, 1, 1) \;,$$

Lorentz transformations are such that

$$\Delta x^{\mathrm{T}} \Lambda^{\mathrm{T}} \eta\, \Lambda\, \Delta x = 1 \quad \text{for all} \quad \Delta x^\mu \;, \qquad \text{or} \qquad \Lambda^{\mathrm{T}} \eta\, \Lambda = 1 \;.$$

Thus, Lorentz transformations are those preserving the 4×4 diagonal matrix η with one eigenvalue of the opposite sign, i.e., the special orthogonal group SO(3,1).

• *Meaning of the interval:* If two events x and x' are such that $(\Delta s)^2 = 0$, then...

• *Spacetime geometry:* Spacetime in special relativity is an affine manifold with a metric $\eta_{\mu\nu}$ with which geometrical quantities can be calculated. For example, the length of a spacelike curve γ between A and B is

$$L = \int_A^B \mathrm{d}\lambda \sqrt{\eta_{\mu\nu}\, \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\, \frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} \;.$$

◇ *Spacetime diagrams:* Representing on a spacetime diagram spacetime subsets and transformations, the relativity of simultaneity, proper time and time dilation, proper length and length contraction.

Reading

Our textbook: Carroll, Chapter 1 up to Section 3; *Other books:* Wald, Ch 2; Schutz, Ch 2.