Aspects of Newtonian Gravity

The Two Main Concepts

• *Newton's law of universal gravity:* The force of gravity on a pointlike object of mass m due to another pointlike mass M a distance r away, or a mass distribution $\rho(\mathbf{r})$ in a volume V, are respectively

$$\mathbf{F}_{\rm g} = -G\,\frac{mM}{r^2}\,\hat{\mathbf{r}}\;, \qquad \mathbf{F}_{\rm g} = -Gm \int_V {\rm d}\mathbf{r}'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,(\mathbf{r}-\mathbf{r}')\;,$$

where $G = 6.67\times 10^{-11}$ N·m^2/kg^2 is Newton's gravitational constant (this is the fundamental constant known with the least precision; it is difficult to measure because in the lab gravity is weak and cannot be screened, and in astrophysics G always appears in the combination GM). This is an instantaneous action at a distance. Alternatively, we can say that $\mathbf{F}_{\rm g} = m\mathbf{g}$, where the gravitational field $\mathbf{g}$ is given by, respectively,

$$\mathbf{g} = -G\,\frac{M}{r^2}\,\hat{\mathbf{r}}\;, \qquad \mathbf{g} = -G \int_V {\rm d}\mathbf{r}'\,\frac{\rho(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|^3}\,(\mathbf{r}-\mathbf{r}')\;.$$

• *Gravitational potential:* Because the force of gravity is conservative, it has a potential $\Phi(\mathbf{r})$ such that $\mathbf{g} = -\nabla\Phi$. In the two situations considered above, respectively, the potential can be written as

$$\Phi(\mathbf{r}) = -G\,\frac{M}{r}\;, \qquad \Phi = -G \int_V {\rm d}\mathbf{r}'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\;,$$

and is a solution of the Poisson equation $\nabla^2\Phi(\mathbf{r}) = 4\pi G\,\rho(\mathbf{r})$, with suitable boundary conditions.

Comments on the Structure of the Theory

• *Remark:* In almost all respects Newton's law of gravity is analogous to electrostatics with Newton's law of gravity replacing Coulomb's law, G replacing $k = 1/4\pi\epsilon_0$ and mass replacing charge. One difference is that only positive masses have been observed. Also, does the analogy extend to time-varying situations?

• *Gravitational multipoles:* When the gravitational field is produced by a mass distribution ρ localized in a region V that is small compared to the distance to the field point $\mathbf{r}$ where we want to find Φ, a good approximation to the potential is obtained by expanding $1/|\mathbf{r}-\mathbf{r}'|$ in powers of $1/r$ and writing

$$\Phi(\mathbf{r}) = -\frac{GM}{r} - \frac{G}{r^3}\sum\nolimits_k r^k D^k - \frac{G}{2r^5}\sum\nolimits_{k,l} Q^{kl}\,r^k r^l - \ldots\;,$$

with the total mass ("monopole moment") M, dipole moment D^k, and quadrupole moment Q^{kl} defined by

$$M := \int_V {\rm d}^3r'\rho(\mathbf{r}')\;, \qquad D^k := \int_V {\rm d}^3r'\rho(\mathbf{r}')\,r'^k\;, \qquad Q^{kl} = \int_V {\rm d}^3r'\rho(\mathbf{r}')\,(3r'^k r'^l - r'^2\delta^{lk})\;.$$

Higher-order multipole moments of the mass distribution can be similarly defined and give rise to terms with higher-order powers of r^{-1} in the expansion of $\Phi(\mathbf{r})$.

• *Structure of spacetime:* In Newtonian physics reality exists in space, a flat, infinite 3D manifold in which matter moves in time. One may view spacetime as a 4D manifold with a preferred set of spatial hypersurfaces t = constant: simultaneity is absolute.

Physical Consequences and Observations

• *Gravitational and inertial mass:* It is an experimental fact that all (small, non-spinning, neutral) bodies fall with the same acceleration when subject only to gravity. In Newtonian physics this is because the gravitational mass $m_{\rm g}$ in $\mathbf{F}_{\rm g} = m_{\rm g}\mathbf{g}$ equals the inertial mass $m_{\rm i}$ in $\mathbf{F}_{\rm net} = m_{\rm i}\mathbf{a}$, so that for all of them $\mathbf{a} = \mathbf{g}$.

• *Freely falling frames:* All test bodies experience the same acceleration when freely falling in a gravitational field, so if we put ourselves in the rest frame of one such object, they will all appear to have zero acceleration. In other words, a gravitational field can be made to disappear by going to an appropriate frame.

• *Principle of equivalence:* Measurements by a stationary observer in a gravitational field are indistinguishable from those by an accelerated observer in the absence of a gravitational field (gravity is like a fictitious or inertial force). More precise formulations of this principle differ depending on the systems being observed (in general it does not hold for particles with internal structure such as spin) and the region in which measurements are carried out (it only holds for regions small compared to the scales over which $\mathbf{g}$ varies).

• *Tidal forces:* If the system subject to the force of gravity is not small compared to the scales over which a gravitational field varies, then tidal forces are an effect that cannot be transformed away by changing reference frame, and there are tests that will detect the presence of the gravitational field. If $\mathbf{r}$ is the deviation vector from a fixed point in a freely falling frame, then the tidal force is given by

$$f^k(\mathbf{r}) = -\sum\nolimits_l m r^l \, \frac{\partial^2\Phi(0)}{\partial r^l \partial r^k} = -mc^2 \sum\nolimits_l R^k{}_{0l0}\, r^l \;,$$

where $R^k{}_{0l0}$ is the tidal force tensor. Notice that if $\rho = 0$ then $\nabla\cdot\mathbf{f} = 0$.

• *Mach's principle:* Ernst Mach proposed that inertial reference frames are determined by the overall mass distribution in the universe. (Think about Newton's bucket thought experiment.)

• *The need for a new theory:* At the beginning of the 20th century there was no compelling reason to believe that Newtonian gravity had to be changed–it predicted a precession for the perihelion of Mercury that differed from the observed one by $43''$/century, but there could have been various reasons. The principle of equivalence does not imply that we need a new theory to replace Newton's theory of gravity, although it may suggest it. The reason we need a new theory is that Newton's is not compatible with special relativity.

Possible Alternative Theories of Gravity

• *First possibilities:* Additional components or modifications to the force have been considered for a long time, and tested. One modification consists in adding a Yukawa-type term to the potential,

$$\Phi(\mathbf{r}) = -\frac{GM}{r}\,(1 + \alpha\, \mathrm{e}^{-r/\lambda}) \;,$$

where α and λ are constants; α is small, while λ must be at least of the order of galactic scales and is related to a graviton mass, a possibility being currently considered as a candidate reason for the cosmological acceleration. Among the other possibilities, one can include additional fields (equally coupled to matter).

• *Gravitomagnetism:* Constructing a relativistic theory as an extension of Newton's can be accomplished in various ways. The most straightforward one may be to do what Maxwell's equations did for the original expression for the electric force in the form of Coulomb's law. This was first proposed by Oliver Heaviside in 1893 and gives the theory now known as gravitomagnetism, with field equations

$$\nabla\cdot\mathbf{g} = -4\pi G\rho \;, \quad \nabla\times\mathbf{g} = -\partial\mathbf{h}/\partial t \;,$$

$$\nabla\cdot\mathbf{h} = 0 \;, \quad \nabla\times\mathbf{h} = -4\pi H\rho\mathbf{v} + (H/G)\,\partial\mathbf{g}/\partial t \;.$$

• *Status of this theory:* The theory defined by the four equations above cannot be a full relativistic theory of gravity, because it does not take into account the self-interaction of the gravitational field—the field has a non-vanishing energy density, which is equivalent to a non-vanishing mass density, and must therefore itself act as a field source. However, in weak-field, slow-motion situations this theory provides a good approximation and predicts new effects due to moving masses—such as the precession of the axis of a spinning mass—which have been observed and, like tidal forces, cannot be transformed away in a freely falling frame.

• *What happened instead:* Motivated by the equivalence principle and by Mach's principle, Einstein developed a theory in which gravity is identified with the geometry of spacetime, and introduced it in 1915. Since the observation 1919 of light deflection by the Sun it is considered to be the best available theory of gravity.

Reading

• *Our textbook:* Carroll, parts of Chapter 1.

• *Other books:* Ohanian & Ruffini, Ch. 1 [recommended]; Hartle, Ch. 3; (Wald; Schutz).