Special Relativity I: Spacetime

Basic Definitions

• *Event, spacetime manifold:* Spacetime is the set of all possible locations of physical phenomena, in space and time. In ordinary approaches it is modeled as a 4-dimensional manifold M, a set of elements called events with suitable sets of coordinate systems used to identify events.

• *Light cones, causality:* Each event p in spacetime has a corresponding past (the set of events that can influence p) and a future (the set of events that p can influence). The boundaries of those sets are the light cone of p, representing events that can be reached moving towards/from p at the speed of light. Knowledge of the pasts/futures or the light cones is equivalent to knowing the causal structure of M.

• *Worldline:* A 1D line in M representing the set of locations of a particle in time. Because particles cannot move faster than light, each point along a worldline must be in the past or future of every other one.

• *Coordinates, space, simultaneity:* Coordinate systems (when not considering gravity and curved spacetime) ordinarily consist of one timelike coordinate t whose value represents the reading of a clock at each location, and three spacelike coordinates $\mathbf{x}$. Space is the set of all events in M with the same value of t; the geometry of such a surface is the usual Euclidean one, and different events in it are called simultaneous.

• *Inertial coordinates:* An inertial coordinate system $\{x^\mu\} = \{(t, \mathbf{x})\}$ is one based on an inertial observer; conventionally, $\mu = 0, 1, 2, 3$. The location (worldline) of that observer is assumed to be at $\mathbf{x} = 0$.

Here we will only consider explicitly Cartesian coordinates. Coordinate system of other types (spherical, cylindrical, etc) are related to Cartesian ones (with the same t axis) by the usual coordinate transformations.

Lorentz Trasformations and Minkowski Spacetime

• *Postulate 1:* (Relativity) The laws of physics are the same in all inertial reference frames. This principle was in effect used in prerelativity physics, but it gained prominence only with special relativity.

• *Postulate 2:* (Speed of light) The speed of light has the same value in all reference frames. This fact is highly non-intuitive, but it follows from the results of the Michelson-Morley experiment.

• *Poincaré and Lorentz trasformations:* Poincaré transformations are the most general linear maps $\Lambda : M \to M$ that take inertial coordinate systems to other inertial coordinate systems, consistently with the speed of light postulate. The simplest ones are translations $x^\mu \mapsto x^\mu + a^\mu$, where the a^μ are constants. The remaining ones, rotations and boosts, are Lorentz trasformations. Rotations can be represented by matrices R such that $\mathbf{x} \mapsto \mathbf{x}' = R\,\mathbf{x}$, as in Euclidean geometry. For example, for a rotation by an angle θ around the z axis,

$$R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} .$$

Boosts are transformations between frames set up by observers with non-zero relative velocities. For example, if observer O' is moving with velocity $\mathbf{v} = v\,\mathbf{i}$ with respect to observer O, then $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, where

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{pmatrix} , \qquad \gamma := (1-\beta^2)^{-1/2} , \qquad \beta := v/c .$$

• *Invariant interval:* For any $\Delta x^\mu = x'^\mu - x^\mu$, all Poincaré transformations leave invariant the expression

$$\Delta s^2 = -c^2 \Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 ,$$

which therefore represents a coordinate independent, physically meaningful quantity characterizing the relationship between two events: it is the proper time elapsed $\Delta\tau = \sqrt{-\Delta s^2}$ if $\Delta s^2 < 0$, or the proper length $L_0 = \sqrt{\Delta s^2}$ if $\Delta s^2 > 0$. Pairs of events for which Δs^2 is negative, zero or positive, respectively, are called timelike, null and spacelike related. Note: We will usually assume that in our units the numerical value of the speed of light is 1; for example, we might be measuring times in seconds and distances in light-seconds.

• *The Lorentz group:* In Euclidean geometry the squared length of a vector $\mathbf{x}$ is given by $\|\mathbf{x}\|^2 := \mathbf{x}\cdot\mathbf{x}$, which can also be written in matrix form as $\mathbf{x}^{\mathrm{T}}\,1\,\mathbf{x}$, and the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ can be obtained from the dot product $\mathbf{x}\cdot\mathbf{y} = \mathbf{x}^{\mathrm{T}}\,1\,\mathbf{y}$, where here "1" is the identity matrix. Since a rotation $\mathbf{x}\mapsto R\,\mathbf{x}$ (under which $\mathbf{x}^{\mathrm{T}}\mapsto\mathbf{x}^{\mathrm{T}}R^{\mathrm{T}}$) does not change lengths or angles, the matrix R must be such that

$$R^{\mathrm{T}}1\,R = R^{\mathrm{T}}R = 1\;, \quad \text{i.e.}\;, \quad R^{\mathrm{T}} = R^{-1}\ (R \text{ is an } \textit{orthogonal} \text{ matrix})\;.$$

If we take the determinant of the equality $R^{\mathrm{T}}1\,R = 1$ ("R preserves the matrix 1") and use $\det R = \det R^{\mathrm{T}}$, we get $(\det R)^2 = 1$. A matrix with $\det R = -1$ would invert the orientation of the axes, so we choose $\det R = 1$ (R is a *special* matrix). Because rotations are 3×3 matrices, we end up with the group SO(3).

The Minkowskian geometry of special relativity is encoded in an analogous but different property of Lorentz transformations. Since the squared interval $\Delta s^2 = \Delta x^\mu\,\eta_{\mu\nu}\,\Delta x^\nu = \Delta x^{\mathrm{T}}\eta\,\Delta x$, where $\eta_{\mu\nu}$ is the matrix

$$\eta = \mathrm{diag}(-1,1,1,1)\;,$$

the Minkowski metric, Lorentz transformations are such that

$$\Delta x^{\mathrm{T}}\Lambda^{\mathrm{T}}\eta\,\Lambda\,\Delta x = 1 \quad \text{for all} \quad \Delta x^\mu\;, \qquad \text{or} \qquad \Lambda^{\mathrm{T}}\eta\,\Lambda = 1\;.$$

Thus, Lorentz transformations are those preserving the 4×4 diagonal matrix η with one eigenvalue of the opposite sign, i.e., the special orthogonal group SO(3,1).

• *Spacetime geometry:* Spacetime in special relativity is an affine manifold with a metric $\eta_{\mu\nu}$ with which geometrical quantities can be calculated. For example, the length of a spacelike curve γ between A and B is

$$L = \int_A^B \mathrm{d}\lambda\,\sqrt{\eta_{\mu\nu}\,\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\,\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}}\;.$$

◇ *Spacetime diagrams:* Representing on a spacetime diagram spacetime subsets and transformations, the relativity of simultaneity, proper time and time dilation, proper length and length contraction.

Tangent Vectors

• *Idea:* Physically, a vector is a quantity with direction and magnitude; mathematicaly, it is an element of a vector space identified by its components in a basis. In Euclidean or Minkowski space displacement vectors can be represented by drawing lines on the manifold itself, and many other types of vectors can be defined using displacements as starting points. In the setting of general relativity, however, the mathematical approach will still hold, but there is no general concept of vector displacements on a manifold and we will need a better approach for relating elements of vector spaces to objects in spacetime.

• *Definition:* A vector at a point $p\in M$ corresponds to a way of taking directional derivatives of functions there. An element $V = \{V^\mu\}$ of a vector space and coordinates $\{x^\mu\}$ can be used to define a directional derivative $V^\mu\partial_\mu f$ of a function f at p. Conversely, a curve γ through x given parametrically by $x^\mu = x^\mu(\lambda)$ defines a derivative along γ, $\mathrm{d}f/\mathrm{d}\lambda = (\mathrm{d}x^\mu/\mathrm{d}\lambda)\,(\partial f/\partial x^\mu)$, so that

$$V^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda\;.$$

◇ *Examples:* The 4-velocity $u^\mu = \mathrm{d}x^\mu/\mathrm{d}\tau$ of a particle; The coordinate vectors $\partial/\partial x^\mu$.

• *Bases and components:* The set of all tangent vectors at p forms the tangent space T_pM. Given any basis $\hat e_{(\mu)}$ for T_pM, the coordinate basis $\{\partial/\partial x^\mu\}$ or any other one, a vector can be decomposed as $V = V^\mu\,\hat e_{(\mu)}$. At a single point, given a basis $\hat e_{(\mu)}$ one can always find coordinates with coordinate lines aligned with the $\hat e_{(\mu)}$, but this is not true in general in any extended region of a manifold.

• *Transformation properties:* From the first definition of vector, we get that when we change coordinates from $\{x^\mu\}$ to $\{x'^\mu\}$, the components change according to

$$V'^\mu = \Lambda^\mu{}_\nu\,V^\nu\;, \quad \text{where} \quad \Lambda^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}\;.$$

Tangent vectors are *contravariant* because they transform with the inverse of the matrix with which the basis elements transform, $\hat e'_{(\mu)} = \hat e_{(\nu)}\,(\partial x^\nu/\partial x'^\mu) = e_{(\nu)}\,(\Lambda^{-1})^\nu{}_\mu$.

Reading

Our textbook: Carroll, Chapter 1 up to Section 4; *Other books:* Wald, Ch 2; Schutz, Ch 2.