

Aspects of Newtonian Gravity

Summary of the Main Facts

- *Newton's law of universal gravity:* The force of gravity on a pointlike object of mass m due to another pointlike mass M a distance r away, or a mass distribution $\rho(\mathbf{r})$ in a volume V , are respectively

$$\mathbf{F}_g = -G \frac{mM}{r^2} \hat{\mathbf{r}}, \quad \mathbf{F}_g = -Gm \int_V d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}'),$$

where $G = 6.67 \times 10^{-11} \text{ N}\cdot\text{m}^2/\text{kg}^2$ is Newton's gravitational constant (this is the fundamental constant known with the least precision; it is difficult to measure because in the lab gravity is weak and cannot be screened, and in astrophysics G always appears in the combination GM). This is an instantaneous action at a distance. Alternatively, we can say that $\mathbf{F}_g = m\mathbf{g}$, where the gravitational field $\mathbf{g}$ is given by, respectively,

$$\mathbf{g} = -G \frac{M}{r^2} \hat{\mathbf{r}}, \quad \mathbf{g} = -G \int_V d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} (\mathbf{r} - \mathbf{r}').$$

- *Gravitational potential:* Because the force of gravity is conservative, it has a potential $\Phi(\mathbf{r})$ such that $\mathbf{g} = -\nabla\Phi$. In the two situations considered above, respectively, the potential can be written as

$$\Phi(\mathbf{r}) = -G \frac{M}{r}, \quad \Phi = -G \int_V d\mathbf{r}' \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|},$$

or it can be found as a solution of the Poisson equation

$$\nabla^2\Phi(\mathbf{r}) = 4\pi G \rho(\mathbf{r}),$$

satisfying the boundary conditions given by the problem under consideration.

- *Gravitational multipoles:* In many situations, when the gravitational field is produced by a mass distribution ρ localized in a region V that is small compared to the distance to the field point $\mathbf{r}$ where we want to find Φ , we can find a good approximation to the potential by expanding $1/|\mathbf{r} - \mathbf{r}'|$ in powers of $1/r$ and writing

$$\Phi(\mathbf{r}) = -\frac{GM}{r} - \frac{G}{r^3} \sum_k r^k D^k - \frac{G}{2r^5} \sum_{k,l} Q^{kl} r^k r^l - \dots,$$

where M is the total mass (or “monopole moment”) of the mass distribution, D^k its dipole moment,

$$M := \int_V d^3r' \rho(\mathbf{r}'), \quad D^k := \int_V d^3r' \rho(\mathbf{r}') r'^k,$$

and Q^{kl} its quadrupole moment,

$$Q^{kl} = \int_V d^3r' \rho(\mathbf{r}') (3r'^k r'^l - r'^2 \delta^{kl}).$$

Higher-order multipole moments of the mass distribution can be similarly defined and give rise to higher-order terms in the expansion of $\Phi(\mathbf{r})$.

- *Remark:* So far, in almost all respects, gravity is exactly analogous to electrostatics with Newton's law of gravity replacing Coulomb's law, G replacing $k = 1/4\pi\epsilon_0$ and mass replacing charge, except that only positive masses have been observed...

Consequences and Observations

- *Gravitational and inertial mass:* It is an experimental fact that all (small, non-spinning, test) bodies fall with the same acceleration when subject only to a gravitational field. This means that the gravitational mass m_g in $\mathbf{F}_g = m_g\mathbf{g}$ and the inertial mass m_i in $\mathbf{F}_{\text{net}} = m_i\mathbf{a}$ are equal, so that for all of them $\mathbf{a} = \mathbf{g}$.

- *Freely falling frames*: Since all bodies experience the same acceleration when freely falling in a gravitational field, if we put ourselves in the rest frame of one such object, they will all appear to have zero acceleration. In other words, a gravitational field can be made to disappear by going to an appropriate frame.
- *Principle of equivalence*: Measurements by a stationary observer in a gravitational field are indistinguishable from those by an accelerated observer in the absence of a gravitational field (gravity is like a fictitious or inertial force). More precise formulations of this principle differ depending on the systems being observed (in general it does not hold for particles with internal structure such as spin) and the region in which measurements are carried out (it only holds for regions small compared to the scales over which $\mathbf{g}$ varies).
- *Tidal forces*: If the system subject to the force of gravity is not small compared to the scales over which a gravitational field varies, then tidal forces are an effect that cannot be transformed away by changing reference frame, and there are tests that will detect the presence of the gravitational field. If $\mathbf{r}$ is the deviation vector from a fixed point in a freely falling frame, then the tidal force is given by

$$f^k(\mathbf{r}) = - \sum_l m r^l \frac{\partial^2 \Phi(0)}{\partial r^l \partial r^k} = -mc^2 \sum_l R^k{}_{0l0} r^l ,$$

where $R^k{}_{0l0}$ is the tidal force tensor. Notice that if $\rho = 0$ then $\nabla \cdot \mathbf{f} = 0$.

- *Mach's principle*: Inertial reference frames are determined by the overall mass distribution in the universe. (Think about Newton's bucket thought experiment.)
- *The need for a new theory*: At the beginning of the 20th century there was no compelling reason to believe that Newtonian gravity had to be changed—it predicted a precession for the perihelion of Mercury that differed from the observed one by 43"/century, but there could have been various reasons. The principle of equivalence does not imply that we need a new theory to replace Newton's theory of gravity, although it may suggest it. The reason we need a new theory is that Newton's is not compatible with special relativity.

Possible Alternative Theories of Gravity

- *First possibilities*: Additional components or modifications to the force have been considered for a long time, and tested. One modification that depends on two parameters (α and λ) consists in adding a Yukawa-type term to the potential,

$$\Phi(\mathbf{r}) = -\frac{GM}{r} e^{-r/\lambda} ,$$

where the constant λ must be at least of the order of galactic scales and is related to a graviton mass, a possibility being currently considered as a candidate reason for the cosmological acceleration. Among the other possibilities, one can include additional fields (equally coupled to matter).

- *Gravitomagnetism*: Constructing a relativistic theory as an extension of Newton's can be accomplished in various ways. The most straightforward one may be to do what Maxwell's equations did for the original expression for the electric force in the form of Coulomb's law. This was first proposed by Oliver Heaviside in 1893 and gives the theory now known as gravitomagnetism, with field equations

$$\begin{aligned} \nabla \cdot \mathbf{g} &= -4\pi G\rho , & \nabla \times \mathbf{g} &= -\partial \mathbf{h} / \partial t , \\ \nabla \cdot \mathbf{h} &= 0 , & \nabla \times \mathbf{h} &= -4\pi H\rho \mathbf{v} + (H/G) \partial \mathbf{g} / \partial t . \end{aligned}$$

- *The problem with this theory*: The theory defined by the four equations above is valid in certain situations (weak-field, slow-motion approximation), but it cannot be a full relativistic theory of gravity, because it does not take into account the self-interaction of the gravitational field.
- *What happened instead*: Motivated by the equivalence principle and by Mach's principle, Einstein developed a theory in which gravity is identified with the geometry of spacetime.
- *What we will do next*: Special relativity, Differential geometry, Einstein's theory.

Reading

- *Our textbook*: Carroll, parts of Chapter 1.
- *Other books*: Ohanian & Ruffini, parts of Chapter 1 [recommended]; (Wald; Hartle; Schutz).